Bargaining between Collaborators of a Stochastic Project

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Abstract

Many business activities require collaboration between firms. The expected surplus generated by the collaboration can evolve stochastically due to changing market conditions, arrival of new information, or discovery of fit between the collaborators. For the project to be implemented, both collaborators have to agree to implement as well as to agree on how the surplus is split. To understand this scenario, I study a model of bilateral bargaining between two symmetric players with a stochastic surplus. The proposer can make repeated offers without commitment to the responder, and they switch roles after some time to allow for counter-offers. The frequency of the counter-offers determines the relative bargaining power at each moment. The paper finds that, when the outside option is irrelevant, players implement the project with efficient delay. Upon implementation, the responder receives a larger share of the surplus than she receives under a static surplus. If the outside option is relevant, the collaboration is inefficient due to a hold-up problem faced by the responder. The model captures different bargaining procedures through varying the frequency of counter-offers, which affects the collaboration outcome. Increasing the frequency of counter-offers improves social efficiency by balancing the bargaining power. Furthermore, bargaining with more frequent counter-offers can lead to Pareto improvements; the proposer benefits too because the increase in efficiency can outweigh the loss of bargaining power. The paper makes a step in understanding why agents may choose to bargain in one way versus another.

Keywords: collaboration, bargaining, optimal stopping, joint decision-making, continuous-time game.

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1 Introduction

Many business activities involve collaborations between firms, such as co-marketing campaign, new product development, joint venture, distribution of products or content, or general outsourcing. Often, neither party has the pricing power over the other. Collaborators have to reach an agreement on whether to implement the project and how the surplus is to be split. This naturally leads to bargaining if firms cannot write complete contracts over all future contingencies.

On the other hand, the expected surplus that the collaboration project generates can evolve over time. This can be due to changing market conditions such as shifting consumer preferences or costs. For example, in 2017, Amazon explored a partnership with DISH Networks to create their own wireless networks. Such venture became attractive as more American have shifted to mobile Internet and smart appliances (Fung, 2017). The expected surplus from a project can also evolve as the arrival of new information resolves uncertainties. Companies often take time to conduct market research before entering a new market. The information collected is used to assess the attractiveness of the market and the feasibility of the entry. In another example, Kraft disputed with Starbucks over a contract breach that enabled Starbucks to expand in the coffee pods market. Starbucks enjoyed success in the pods market during their three years of negotiation. The settlement resulted in Starbucks paying $2.75 million to Kraft, which could be different had the market performed differently (Shonk 2018). The surplus can also evolve as partners discover their fit for collaboration. Apple and IBM began working with each other in 2012 to find "commonalities in opportunities and challenges", which led to a partnership to create enterprise apps in 2014. In another example, after the successes of many smaller scale collaborations, Red Bull and GoPro signed an exclusive global content partnership in 2016. When asked about the partnership, the CEO of GoPro described the two brands as "extremely compatible and collaborative", and stated that "the feedback [GoPro] get from [Red Bull] is phenomenal." (Beer, 2016) Last but not least, a common tactic in negotiation is to work with the opponent in finding new "win-win" solutions (see, e.g. Bazerman et al. 1985 and De Dreu et al. 2000). The result of such effort is uncertain and impacts the total surplus. These examples illustrate different ways that the surplus from a project can evolve over time. Sovereign debt restructuring is another common example.

If the return of a project evolves stochastically, a decision maker may want to delay implementing the project until the surplus is sufficiently high. However, it is less clear what happens if two parties are required to implement the project. In order for the project to be implemented, both firms have to agree to implement at that moment, as well as agreeing on how the surplus is to be split. The timing of the project has to be the equilibrium outcome of a bargaining game, instead of the solution to a single-agent optimization problem.

This paper investigates whether and when collaborators reach an agreement to implement the project, and how they agree to split the surplus. To do so, I present a model of bilateral bargaining with a stochastic value in continuous time. The total surplus follows a Brownian motion, which can be seen as a generalization from the motivating examples. Two symmetric players bargain in a pseudo-alternating-offers fashion. At each moment, one player is the proposer and the other player is the responder. If the proposer wants to implement the project, she can propose a split to the responder. If the proposer makes an offer, the responder then decides whether to accept, wait, or quit and take the outside option. Players switch their roles after some time at the arrival of a Poisson process, so the responder becomes the new proposer and make counter-offers.[1]

A common feature of sequential bargaining games is that the bargaining power is determined by who gets to make the offer and when. In this paper, bargaining power is governed by the arrival rate of the role switch between the proposer and the responder. I refer to this arrival rate as the frequency of counter-offers. If the counter-offers are more frequent, the two parties are more balanced in their power. If the opportunity for counter-offers arrives less frequently, the bargaining power is more unbalanced in favor of the current proposer,

[1]This model of bargaining can also be thought of as an alternating-offers model, with no commitment on the previous offer before the next counter-offer arrives.

because the responder has to wait longer if she wants to wait till she can make her own offers.

One difficulty in modelling bargaining as a sequential game is that the outcome of the game is determined by the choice of the extensive form, but real negotiations could be undertaken in a variety of procedures. In this paper, varying the frequency of counter-offers has the effect of varying the procedures by which players bargain. As the frequency becomes infinitely low, the game approaches the repeated-offers paradigms of Fudenberg et al. (1985) and Gul et al. (1986), where one player makes all the offers. As the frequency becomes infinitely high, the game approaches the continuous-time limit of the alternating-offers game by Rubinstein (1982). Intermediate levels thus captures a class of symmetric bargaining procedures between the two extremes. This allows one to analyze the bargaining outcome under a class of bargaining procedures in a simple way.

The paper shows that one bargaining procedure can be "better" than another. Changing the bargaining procedure can lead to a Pareto improvement by making the timing of collaboration more efficient. Bargaining with more frequent counter-offers can Pareto dominate bargaining under a lower frequency, suggesting that both parties and any social planner may prefer to bargain in a more balanced fashion by allowing for more counter-offers.

The model produces an unique equilibrium outcome under symmetric and stationary strategies. When the outside option is non-positive so that quitting is not relevant, players delay agreement until it is socially efficient to implement the project. Upon agreement, the responder receives a larger share of the surplus than the share she receives if the surplus size is fixed.

When the outside option is strictly positive, quitting is a relevant option. The responder quits when the total surplus drops to a sufficiently low level. In this case, the collaboration is no longer efficient. Players reach agreement or quit negotiation too soon compared to the socially optimal levels. Such inefficiency is a result of the hold-up problem faced by the responder. Waiting for a higher surplus can be interpreted as a relationship-specific investment. The cost of delaying taking the outside option is the discounting of the outside option. The return of that investment is determined by future bargaining. Thus the responder, who has less bargaining power, under-invests and quits too early. This also leads to a premature agreement to implement the project when the surplus is high. This hold-up problem can be mitigated with more frequent counter-offers. A higher frequency of counter-offers balances the bargaining power, which leads to more efficient agreement and quitting decisions. Increasing the speed of counter-offering not only redistribute the ex-ante utility, but also expand the total welfare.

The proposer can profit from a more balanced bargaining procedure if the expansion of total welfare dominates the redistribution of welfare. Particularly, when the initial size of the surplus is not too big or too small, the bargaining outcome under a low frequency of counter-offering can be Pareto dominated by the outcome under a high frequency of counter-offering. This phenomenon does not happen if the size of pie is fixed or if quitting irrelevant; those cases do not exhibit the hold-up problem, so varying the frequency of counter-offers does not affect total ex-ante utility, only the distribution of that utility.

The paper is organized as follows. After literature review, Section 2 present the model. Section 3 presents the equilibrium outcome when quitting is irrelevant, and shows its efficiency. Section 4 solves the case with quitting and shows the effect of counter-offers and bargaining power on social and Pareto efficiency. Section 5 concludes the paper.

1.1 Literature Review

The paper expands on bargaining literature with stochastic payoffs or stochastic proposing order. Merlo and Wilson (1995, 1998) present a general model of stochastic bargaining games and proves the uniqueness of a stationary payoff when there is no outside option. Cripps (1997) studies alternating bargaining with value that follows geometric Brownian motion. Furusawa and Wen (2003) studies the case of stochastic disagreement payoffs and that the proposer can delay proposing. In Hanazono and Watanabe (2016), players receive private signals on an i.i.d stochastic pie.

Another stream of literature expands the alternating-offers model of Rubinstein (1982) to allow for a random order of proposers determined by a homogeneous Markov process, see e.g. Binmore (1987), Muthoo (1999), and Houba (2008). Coles and Muthoo (2003) allows the surplus to be deterministic but non-stationary. Yildiz (2004) and Simsek and Yildiz (2014) studies the case of uncommon prior on the recognition process, and players update their beliefs about bargaining power over time. This paper studies bargaining in continuous time and allows both stochastic surplus and random order of proposing. The paper highlights the relationship between frequency of counter-offers and the efficiency of bargaining when quitting is possible, which is a novel discovery in the literature.

Daley and Green (2017) and Ortner (2017) study stochastic bargaining in continuous time. In Daley and Green (2017), the buyer receives sequential signals on quality of the seller's asset. In Ortner (2017), the surplus is fixed, but the bargaining power is public and evolves over time. Both players only allow one player to make offers, whereas this paper allows for counter-offers.

The collaborators in this paper end negotiation prematurely due to a holdup problem similar to those found in Rogerson (1992), Tirole (1986, 1988), and Williamson (1975). A conceptually related paper is Frankel (1998), in which players can exert effort to expand the size of pie, but sometimes under-invest due to the hold-up problem. The paper is also related to models of stopping problems with stochastic payoffs. Earlier works have examined the cases of R&D funding (e.g., Roberts and Weitzman 1981), options (e.g., Dixit and Pindyck 1994), and consumer search (e.g., Branco et al. 2010). Wilson (2001), Compte and Jehiel (2010), Cho and Matsui (2013), and Kamada and Muto (2015) consider multi-agent search problems in which a unanimous agreement is required to stop. They show that the limiting case of the search models relate to Nash bargaining solution, but do not allow explicit transfer of utility between the players. In this paper, stopping also requires an agreement between players, but they are allows to make offers to facilitate such agreement.

2 The Model

The game is in continuous time with an infinite horizon. Two players, i and j, bargain over how to split the return of a project with a stochastic expected surplus x_t. Players are risk-neutral so that they only care about the expected return. The expected return of the project x_t is observable to both players and is assumed to follow a Brownian motion $dx_t = \sigma dW_t$ with volatility $\sigma \geq 0$ and initial position x_0, where W_t is a Wiener process.

The order of movement at time t is determined by a recognition process $f_t \in K$. We call player i the *Proposer* at time t if $f_t = i$ and the *Responder* at time t if $f_t \neq i$. At each moment, the Proposer can propose a split of the cake x_t and the Responder decides whether to accept the proposal. The roles are switched at the arrival of a Poisson process. Assuming WLOG that $f_0 = 1$, then $f_t = f_0 + N(t) \bmod 2$, where $\{N(t), t \geq 0\}$ is a Poisson counting process with rate λ. If there is no arrival in a time interval T, then the roles remain the same and the same Proposer can make repeated offers at each time $t \in T$. Thus λ captures the speed at which counter-offers happen, and $N(t)$ captures the number of times that offers have been countered. Let (Σ, $\mathcal{F}$, P) be the probability space that supports the Wiener process W_t and the Poisson process N_t, and $\mathrm{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ be the filtration process satisfying the usual assumptions.

The game is played as follows. At time t, the cake size x_t and the identity of the Proposer f_t are realized. Upon the realization, the Proposer can choose to proposes a split $s_t \in R^2_+$ such that $s_1 + s_2 = x_t$ so no waste is allowed. Let $p_t = (s_t)_{j\neq f_t}$ denote the amount offered to the Responder, and denote $p_t = -\infty$ if the Proposer does not make an offer, since making an unacceptable offer is equivalent to not making one. Given an offer p_t, the Responder chooses whether to accept or reject. If the offer is accepted, players implement the project and players split the surplus as agreed. If the offer is rejected, the Responder chooses whether to continue or to quit. If the Responder chooses to quit, the game ends and both players receive their outside options. The game continues with new realizations of x_t and f_t until an offer is accepted or a player quits. Figure 1 illustrates the game graphically.

Figure 1

Players are symmetric with a discount rate r and an outside option of size ω. Upon reaching an agreement, the Proposer receives an expected utility of $x_t - p_t$ and the Responder receives an expected utility of p_t. If the Responder quits, then each player receive her outside option ω.

Let $a_t \in \{0, 1\}$ be an indicator function for whether the Responder agrees to the Proposer's offer at time t, and $q_t \in \{0, 1\}$ be an indicator function for whether Responder quits at time t. Then the game ends at $\tau = \inf\{t | a_t = 1 \text{ or } q_t = 1\}$. The expected utility of player i at time t then is defined as:

$$u_{i,t} = e^{-r(\tau - t)}[\mathbb{1}\{f_\tau = i\}(1 - p_\tau) + \mathbb{1}\{f_\tau \neq i\}p_\tau]$$

if players reach agreement at time τ, i.e., $a_\tau = 1$. And the expected utility of player i at time t is:

$$u_{i,t} = e^{-r(\tau - t)}\omega$$

if a player quits at time τ, i.e., $a_\tau = 0$ and $q_\tau = 1$. Players receive zero utility if players never reach an agreement or quit, i.e., $\tau = \infty$ or $(a_\tau = 0 \ \wedge \ q_\tau = 0)$. I refer to an outside option as *irrelevant* if $\omega \leq 0$, since it can never be taken in equilibrium. The paper refers to an outside option as *relevant* if ω is strictly positive.

Because the players are symmetric and both the surplus and the recognition processes are stationary, the paper focuses on equilibrium outcomes that are symmetric and stationary. Equilibrium strategies depend on the current size of the surplus and the roles of each player, but not on time or identity of the players.

Definition 1. *A strategy profile is an equilibrium if it satisfies subgame-perfection*[2] *and can be described by:*

1. *A proposing strategy* $p_t = p(x_t) : \mathcal{R} \mapsto \mathcal{R}$ *for the Proposer at time* t.
2. *An agreement strategy* $a_t = a(x_t, p_t) : \mathcal{R}^2 \mapsto \{0, 1\}$ *for the Responder at time* t.
3. *A quitting strategy* $q_t = q(x_t) : \mathcal{R} \mapsto \{0, 1\}$ *for the Responder at time* t.

An equilibrium outcome can be described by $\{U(x), V(x), A, Q\}$, where $U(x)$ is the value function of the Proposer in state x, $V(x)$ is the value function of the Responder in state x, $A = \{x \mid a(x, p(x)) = 1\}$ is the region in which players reach agreement, and $Q = \{x \mid q(x) = 1\}$ is the region in which the Responder quits.[3]

Only the Responder is allowed to quit in this game. I argue that this is innocuous. Allowing the Proposer to also quit adds trivial equilibria in which players quit at the same time.[4] Intuitively, the Responder has a weaker position and thus a stronger incentive to quit.

[2] In continuous-time games, a strategy profile may not produce a well-defined outcome. See Simon and Stinchcombe (1989). When checking for profitable deviations, only strategy profiles such that τ is a measurable function from Σ to R^+ is considered.

[3] By the definition of utility, players receive payoffs of 0 if they neither reach agreement or quit at τ. This effectively restrict A and Q to be closed sets. Without the restriction to closed sets, one can create alternative equilibrium outcomes with agreement region $A \setminus Z_1$ and quitting region $Q \setminus Z_2$, where Z_1 and Z_2 are sets of measure 0. See Ortner (2017) for a similar restriction.

[4] If the quitting decisions are made simultaneously, quitting is always (weakly) optimal if the opponent is quitting at the same time. So quitting at any state can be supported in an equilibrium. Sequential order does not eliminate this problem. For example, take any random threshold $\underline{x} < \omega$, and let both players quit

I show that the Proposer always has a (weakly) higher continuation value in equilibrium. Thus, any equilibrium of this game must also be an equilibrium if the Proposer is allowed to quit. In Appendix C, I solve a discrete-time analog of the game in which both players are allowed to quit, with the refinement that players quit if and only if quitting is strictly preferred. I show that as the length of each period approaches 0, the equilibrium outcomes must converge to the continuous-time outcome when the Responder can quit. Thus not allowing the Proposer to quit is without loss in that sense.

If $x \in A$, then the Proposer offers the Responder $p(x)$ and the Responder accepts immediately, so $U(x) = 1 - p(x)$ and $V(x) = p(x)$. If $x \in Q$, then the Responder quits immediately, so $U(x) = V(x) = \omega$. If $x \in \mathcal{R} \setminus (\mathcal{A} \cup \mathcal{Q})$, then the Responder rejects the offer but continues to wait. In this case, one can show that the value functions must satisfy:

$$\begin{aligned}(r+\lambda)U(x) &= \frac{\sigma^2}{2}U''(x) + \lambda V(x) \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x)\end{aligned} \tag{1}$$

or

$$\begin{aligned}(U+V)(x) &= \frac{\sigma^2}{2r}(U+V)''(x) \\ (U-V)(x) &= \frac{\sigma^2}{2r+4\lambda}(U-V)''(x)\end{aligned} \tag{2}$$

The function $(U+V)(x)$ is the sum of both players' expected utilities in state x and captures the social value. Through $(U+V)(x)$ we can examine whether an outcome is efficient. The function $(U-V)(x)$ is the difference in the utilities and captures the advantage of being the Proposer. The full derivation of these equations are in Appendix A.

In equilibrium, the sum of the players' value functions have to exceed the current surplus size and exceed the sum of the outside options. Intuitively, if the sum of their utilities by

for $x < \underline{x}$. Then there's not profitable deviation in these states regardless the order of quitting. Because the state evolves continuously, the later mover is indifferent because the opponent will quit in the "next instant" regardless. The state x_t cannot jump out of the quitting region. Thus any choice of $\underline{x} < \omega$ can be supported in an equilibrium in this trivial fashion.

following their equilibrium strategies is less than what is available right now, then they can profitably deviate by splitting the current surplus or taking the outside options.

Lemma 1. *In equilibrium,* $(U+V)(x) \geq x$ *for all* $x \in \mathcal{R}$.

2.1 Examples of Stochastic Surplus

The model uses a Brownian motion to capture the idea of a stochastic surplus in tractable way. Below I present a few micro motivations for a stochastic surplus, in which Brownian motion can be derived as a limiting case.

A hypothetical firm is considering whether to enter into a market. To operate in that market, the firm has to collaborate with a local partner.

Evolving Consumer Preference The potential channel profit depends on consumer preference, which can change over time. For example, consider a Hotelling line of length l. A mass of consumers is located at $z_t \in [0, l]$ at time t, and z_t takes a random walk with reflecting boundaries at 0 and l. There exists a competitive fringe at location 0 with price equal to 0, and the firm can choose to enter at location l. Consumers buy one and only one product. If the firm enters at time t, the highest amount it can charge to make a sale is $p_t = 2z_t - l$, which is a Brownian motion with reflecting boundaries at l and $-l$. As $l \to \infty$, the profit from entry approaches a Brownian motion without boundary.

Learning about Preference The product provides a value of v_t to consumers, which itself can change over time with a random walk of variance σ^2. The firm does not observe the true value of v_t, and learn through signals obtained from market research. The firm has a normal prior with mean $\hat{v}_0$ and variance $\hat{\rho}_0$. Each moment, the firm receives a signal of v_t plus a normal error with variance η^2, and updates the posterior mean $\hat{v}_t$ and variance $\hat{\rho}_t$ using Bayes' rule. The signal S_t accumulates as $dS_t = v_t dt + \eta dW_t$, where W_t is a Wiener process. By the Kalman-Bucy filter (See Ruymgaart and Soong 1988, Ch.4), the posterior mean $\hat{v}_t$ follows $d\hat{v}_t = (\hat{\rho}_t/\eta)dB_t$ for some Wiener process B_t, and posterior variance follows $\frac{d\hat{\rho}_t}{dt} = -\hat{\rho}_t^2/\eta^2 + \sigma^2$. The posterior variance $\hat{\rho}_t$ approaches $\sigma\eta$ asymptotically over time. If

$\hat{\rho}_0 = \sigma\eta$, then $\hat{v}_t$ is a Brownian motion with variance $\sigma^2\eta^2$.

Matching with Collaborator The success of the entry depends on the match between the firm and the local partner. There are a mass of qualities important for the success of the product. The channel provides more value to consumers if the characteristics of the two collaborators match. The two companies discover their match over time as they explore the potential entry. With equal chance, the collaborators can match on a quality, which provides value $z_t = +\sigma\sqrt{dt}$ to consumers, or do not match, which provides value $-\sigma\sqrt{dt}$, where dt is the length of the each period (or size of each attribute). So $\mathbb{E}[z_t] = 0$ and $Var[z_t] = \sigma^2 dt$. The expected value of the product after observing t attributes then can be written as $x_t = x_0 + \sum_0^t z_s$, where x_0 is the expected value of the product prior to their communication. As the mass of total attributes approach infinity and dt approaches 0, the expected product value x_t becomes a Brownian motion.

2.2 Frequency of Counter-Offers and Bargaining Power

The arrival rate λ captures the frequency by which offers are countered. When the event arrives, the Proposer becomes the Responder, and the Responder becomes the Proposer and can make counter-offers. The players remain in their new roles until the next counter-offer happens. Varying the parameter λ is analogous to varying the extensive-form by which the players bargain. A game with a higher λ features more frequent counter-offers. If $\lambda \to \infty$, the game approaches the continuous-time limit of the alternating-offers paradigm of Rubinstein (1982). On the other hand, as $\lambda \to 0$, all offers are made by the first Proposer (in probability). Thus the game approaches the repeated-offers paradigm of Fudenberg et al. (1985) and Gul et al. (1986) without asymmetric information.

As in other sequential bargaining game, bargaining power between two symmetric agents is determined by who makes the proposal and when; hence, in this game, the choice of λ. This point can be illustrated by examining the static case. Assume $\sigma = 0$ so that the cake size is fixed over time. Also assume $x_0 > \omega = 0$, so the outside option is irrelevant.

This game with a static surplus has an unique equilibrium outcome. Plugging $\sigma = 0$ into equations (1) shows that the expected utility for the Responder if she rejects the first offer is $\frac{\lambda}{r+\lambda}U(x_0)$. Thus the Responder accepts if and only if $p(x_0) \geq \frac{\lambda}{r+\lambda}U(x_0)$. Then the Proposer offers $p_0 = \frac{\lambda}{r+\lambda}U(x_0)$. Since $U(x_0) + V(x_0) = x_0$, this implies that the Proposer gets $\frac{r+\lambda}{r+2\lambda}x_0$, and the Responder gets $\frac{\lambda}{r+2\lambda}x_0$. As λ decreases, the Proposer gets a larger share of the pie. Less counter-offers translates to more bargaining power to the Proposer. Conversely, as λ increases towards infinity, allowing more frequent counter-offers, the Proposer loses her advantage and the split becomes more even. It is worth noting that this equilibrium is analogous to the symmetric equilibrium from Rubinstein (1982). Particularly, if we define $\delta = \frac{\lambda}{r+\lambda}$, then the Proposer's share is $\frac{1}{1+\delta}$ and the Responder's share is $\frac{\delta}{1+\delta}$, as in Rubinstein (1982). This equivalence no longer holds when the size of cake is stochastic.

To model bargaining as a sequential game, one faces the problem of choosin which extensive form to use. Fudenberg et al. (1986) point out two issues, "First, because the results depend on the extensive form, one needs to argue that the chosen specification is...a good approximation to the extensive forms actually played. Second, even if one particular extensive form were used in almost all bargaining, the analysis is incomplete because it has not...begun to address the questions of why that extensive form is used." This view is echoed in Sutton (1986). Modelling the bargaining procedure with λ addresses these two issues. First, it allows us to examine how the equilibrium outcome is affected by a class of bargaining procedures. Second, it provides a meaningful comparison between different procedures. Section 4 shows that, when the outside option is positive, the bargaining outcome under a higher λ can dominate the bargaining outcome under a lower λ in both Pareto efficiency and social efficiency, by mitigating of the Responder's hold-up problem. In such a case, both players and any social planner should prefer to bargain with more frequent counter-offers.

3 Irrelevant Outside Option

This section examines what happens if quitting is not a feasible action. Assume that the outside option is irrelevant, or $\omega \leq 0$. Since the payoff from bargaining indefinitely is zero, and at any point in time, there is always a positive probability that the cake size becomes positive in the future, the expected utility from continuing to bargain is always positive. Thus for irrelevant outside options, we can ignore the quitting action and assume that players remain in the game until they reach an agreement.

3.1 Socially Efficient Outcome

As a benchmark, we first find what the socially efficient outcome is. It is socially optimal to delay the project when the surplus is small, because of the potential of a higher surplus in the future. This is an optimal stopping problem (See, e.g., Dixit 1993) with discount rate r. The optimal decision follows a threshold $\overline{x_s}$. The social planner implements the project if the surplus is equal or above the threshold, and wait otherwise.

One can show that the socially efficient threshold is $\overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$ and the value function of the social planner is $W_s(x) = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}x-1}$. See Appendix A.1 for details.

3.2 Equilibrium Outcome

The following Lemma shows that in equilibrium, players reach agreement once the surplus reaches above a threshold, similar to the social planner's stopping behavior. It's easy to see that players have to reach agreement at some point. When the surplus is high, they can divide up the surplus and both be better off than bargaining indefinitely. If they reach agreement for a state x_l, one can show that they must also reach agreement for any state x_h that is higher than x_l. Conditional on agreeing in state x_l, the socially optimal decision for surplus x_h is to implement the project immediately. Thus x_h is both an upper bound and a lower bound (per Lemma 1) on the sum of players' value functions at x_h. Intuitively, if players are willing to implement the project at x_l, then at least one of the players wants to implement the project when the surplus is higher. Since players can transfer utility through bargaining, the player that wants to implement the project can offer the other player enough so that the project can be implemented.

Lemma 2. *$\exists\, \overline{x} \geq 0$ s.t. $A = [\overline{x}, \infty)$ in equilibrium.*

To characterize an equilibrium outcome, we only need to solve for the agreement threshold, $\overline{x}$, and the value functions, $U(x)$ and $V(x)$. Given an agreement threshold $\overline{x}$, player implement the project when the surplus reaches that threshold.

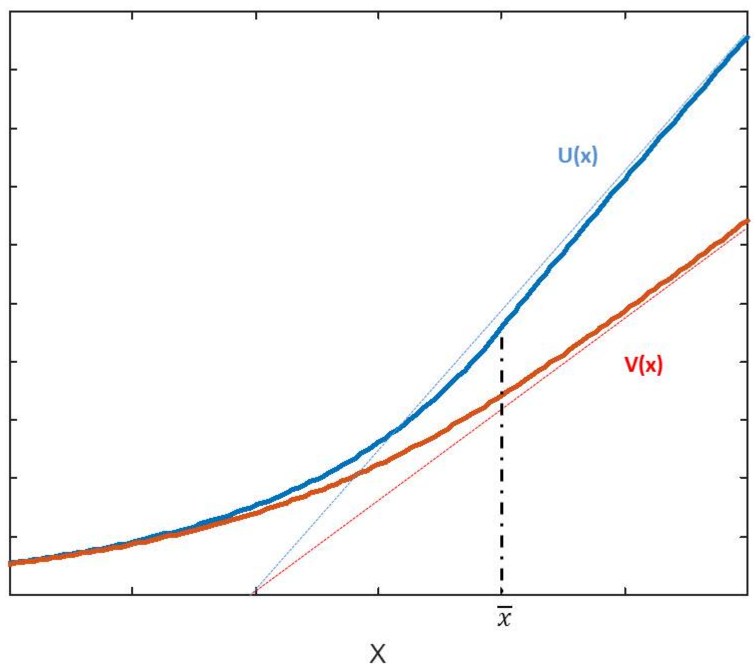
Proposition 1 shows that the agreement threshold in equilibrium must be equal to the socially efficient threshold. Thus the sum of players' equilibrium value functions must also equal to the social planner's value function. The intuition is that, if the outcome is inefficient, then the two players can "coordinate" a profitable deviation. If the agreement threshold is not socially optimal, then there exist a different threshold that improves total welfare, so at least one player must benefit from such deviation. Bargaining then allows that player to transfer some of that efficiency gain to the other player, who must accept given subgame-perfection. Thus both players benefit from such a deviation. The only agreement threshold that players cannot mutually benefit from deviating is the socially optimal one. Thus any equilibrium outcome must experience the same delay as a social planner would: the Responder rejects the Proposer's offer when the surplus is smaller than $\overline{x_s}$, and accepts the offer when the surplus is larger.

Knowing the agreement threshold, one can find proposal rules that implements such threshold. For a proposal rule, both players have to be willing to stop exactly at the socially efficient threshold. The proposal rule is unique in the agreement region, which implies that the equilibrium outcome is unique. The proposal rule can be different for the non-agreement region, but such multiplicity do not affect the equilibrium outcome. If an offer is rejected, the size of that offer becomes irrelevant.

Proposition 1 (Irrelevant Outside Option with $\omega \leq 0$)**.** *There exists an unique equilibrium* *outcome. The agreement threshold is* $\overline{x} = \overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$. *The Proposer has a higher utility than the Responder for all* x. *Upon agreement, the Responder receives strictly more than* $\frac{\lambda}{r+2\lambda}$ *share of the surplus, which is what the Responder receives if the surplus is static.*

Closed-form solutions of the value functions $U(x)$ and $V(x)$ are presented in Appendix A.2. Figure 2 depicts the value functions and the trading threshold graphically. The Proposer's value function is always strictly above the Responder's, illustrating the advantage of being the Proposer.

Figure 2: Equilibrium Outcome



Another result from Proposition 1 is that the Responder gets a bigger share of the surplus than she gets from a static surplus. When the project is implemented, the Responder receives strictly more than $\frac{\lambda}{r+2\lambda}$ share of the surplus, which is her share of the cake in the static model in Section 2.2. Thus the advantage of being the Proposer is smaller than when the surplus is static. In Figure 2, the dotted lines represent the shares that players would receive under a fixed surplus. In equilibrium, the Responder receives strictly above the static share. Also, the equilibrium split approaches the static split as $x \to \infty$.

The stochastic nature of the project provides more bargaining power to the Responder because the Responder can unilaterally delay project implementation. Because the Responder gets a smaller share of the surplus than the Proposer, she incurs less cost from delaying the project. The Responder then has more incentive to wait. To see this, suppose that the only available split is their static share, which is $\frac{r+\lambda}{r+2\lambda}x$ for the Proposer and $\frac{\lambda}{r+2\lambda}x$ for the Responder. Player then decide individually at each point, whether they agree to such a split. One can show that the Responder must have a higher agreement threshold. Since both players have to agree for the project to be implemented, the Responder can unilaterally delay the project. The Proposer thus has to offer more than the static share to the Responder in order to encourage the Responder to agree earlier.

Figure 3: Responder delay agreement for fixed static share

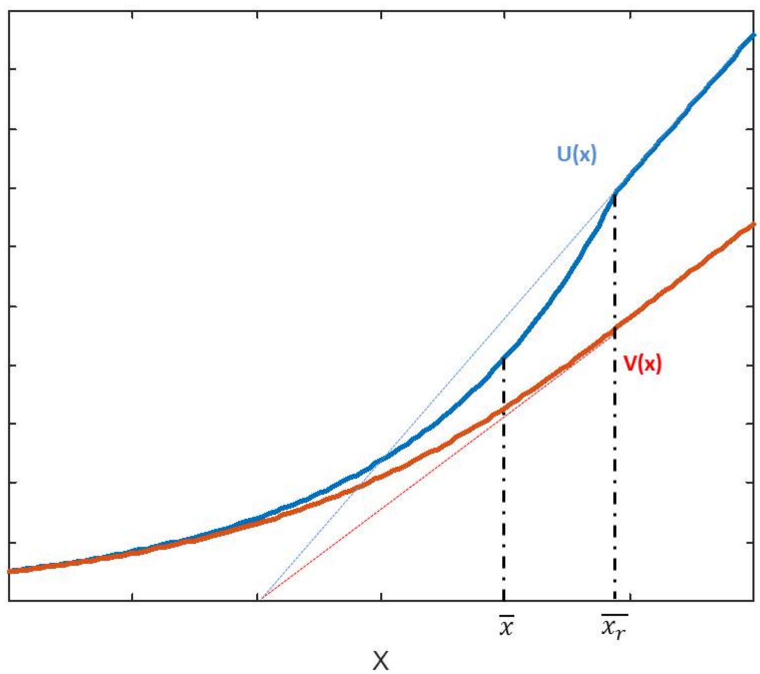


Figure 3 illustrates this example. The threshold $\overline{x}$ is the equilibrium threshold with bargaining, and the threshold $\overline{x_r}$ is the Responder's stopping threshold if she can only get $\frac{r+\lambda}{r+2\lambda}x$. For states between $\overline{x}$ and $\overline{x_r}$, the sum of players' value functions is smaller than the size of the surplus. Thus the Proposer prefers to offer more to the Responder in order to reach an agreement.

Regardless of the starting position, the equilibrium agreement threshold does not depend on λ. Thus the choice of λ does not affect the efficiency but only affects how the total utility is distributed. One can easily verify that $U(x)$ decreases with λ and $V(x)$ increases with λ for all x_0. As $\lambda \to \infty$, each player receives half of the social value. As $\lambda \to 0$, the Proposer receives the whole social surplus. Thus a slower pace of counter-offers gives more power to the Proposer, and a faster pace of counter-offers makes the bargaining power more balanced, same as in the static case.

Corollary 2 (Comparative Statics w.r.t λ)**.** *The equilibrium outcome is socially efficient for all λ. The Proposer's ex-ante utility strictly increases with λ, and the Responder's ex-ante utility strictly decreases with λ.*

4 Relevant Outside Option and the Hold-up Problem

Suppose now that the outside option is relevant, or $\omega > 0$, so that the Responder has an incentive to quit if her continuation value is low. In this case, the outcome is no longer socially efficient. More importantly, the level of efficiency is determined by the frequency of counter-offers, λ. Furthermore, the equilibrium outcome under a higher λ can Pareto dominate the outcome under a lower λ, suggesting that there can be mutual gain by allowing for more counter-offers.

4.1 Socially Efficient Outcome

A social planner with discount rate r and outside option 2ω choose at each moment whether to implement the project, wait, or take the outside option. The social planner implements the project if the surplus reaches above the threshold $\overline{x_s}$, and takes the outside option if the surplus reaches below the threshold $\underline{x_s}$. Solving the social planner's optimal stopping problem shows that the socially efficient thresholds are

$$\overline{x_s} = \sqrt{\frac{\sigma^2}{2r} + 4\omega^2} \quad \text{and} \quad \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\Big(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\Big)$$

The social value function for continuation states $\underline{x_s} < x < \overline{x_s}$ is:

$$W_s(x) = \frac{1}{2}\Big(\overline{x_s} + \sqrt{\frac{\sigma^2}{2r}}\Big)e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} + \frac{1}{2}\Big(\overline{x_s} - \sqrt{\frac{\sigma^2}{2r}}\Big)e^{-\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} \tag{3}$$

See Appendix A.2 for more details.

4.2 Equilibrium Outcome

Same as the irrelevant outside option case, there exists an agreement threshold $\overline{x}$. Players reach agreement immediately if the surplus is larger than the threshold. The agreement threshold has to be lower than the socially efficient threshold $\overline{x_s}$, otherwise players can profitably deviate.

Lemma 3. *In equilibrium, $\exists$ an agreement threshold $\overline{x}$ such that $\overline{x_s} \geq \overline{x} \geq 2\omega$ and $A = [\overline{x}, \infty)$.*

Because quitting is a relevant choice now, to characterize any equilibrium outcome we have to specify the states in which the Responder quits. The following result says those states can be described by a quitting threshold $\underline{x}$. The negotiation breaks down when the surplus drop below such a threshold. The quitting threshold has to be higher than the socially efficient threshold $\underline{x_s}$.

Lemma 4. *In equilibrium, $\exists$ a quitting threshold $\underline{x}$ such that $\underline{x} \geq \underline{x}_s$ and $Q = (-\infty, \underline{x}]$.*

An equilibrium outcome can thus be described by $\overline{x}$, $\underline{x}$, $U(x)$ and $V(x)$. When quitting is not a relevant option, players implement the project at the social efficient threshold in equilibrium, as shown in Proposition 1. This is no longer true when the outside option is relevant. In equilibrium, players quit and agree both earlier than what is socially optimal. The following Proposition describes the unique outcome. In equilibrium, $V(x)$ has to be smooth over all x, and $U(x)$ has to be smooth for all x other than $\underline{x}$. Together they imply an unique outcome. See Appendix B for the proof.

Proposition 3 (Relevant Outside Option with $\omega > 0$)**.**

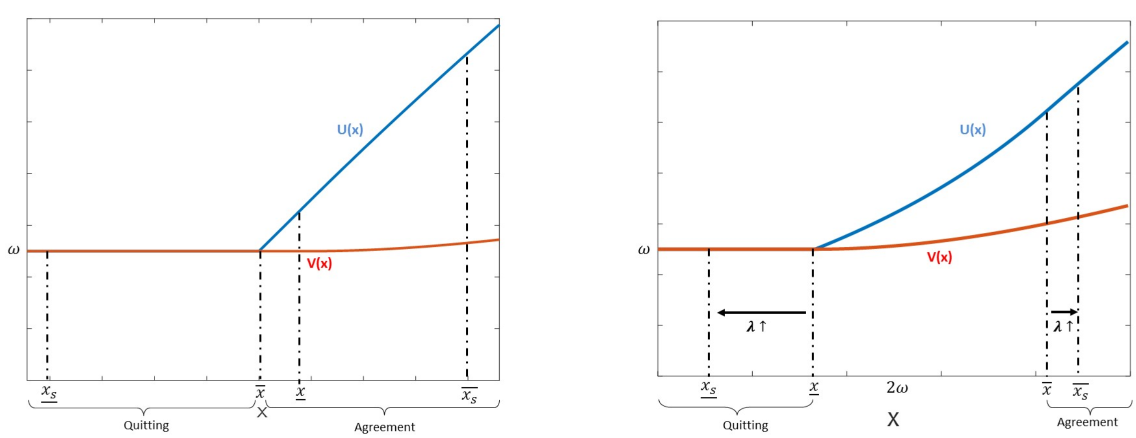
1. *There exists an unique equilibrium outcome with agreement threshold $\overline{x} < \overline{x_s}$ and quitting threshold $\underline{x} > \underline{x_s}$. The Proposer has a weakly higher utility than the Responder for all x.*

2. *Let $\underline{\lambda} = \frac{4r^2\omega^2+\sqrt{16r^4\omega^4+8\sigma^2r^3\omega^2}}{2\sigma^2}$:*

   - *If $\lambda \leq \underline{\lambda}$, then $\underline{x} \geq \overline{x} = 2\omega$. Games ends immediately for all initial positions.*
   - *If $\lambda > \underline{\lambda}$, then $\overline{x} > \underline{x}$. The agreement threshold $\overline{x}$ strictly increases in λ and the quitting threshold $\underline{x}$ strictly decreases in λ.*

The closed-form expression of players' value functions can be found in Appendix A.2. Figure 4 graphically depicts the unique equilibrium outcome, and compare it to the socially efficient thresholds.

Figure 4: Equilibrium Outcome with Relevant Outside Option



(a) $\lambda \leq \underline{\lambda}$ (b) $\lambda > \underline{\lambda}$

The Responder quits earlier than what is socially optimal because she faces a hold-up problem from her lack of bargaining power. When deciding between quitting now and waiting a little bit longer, her cost of continuation is the discounting of the outside option, and her benefit of continuation is the potential of a higher surplus from future project implementation. However, her benefit from continuation depends on future bargaining, and her lack of bargaining power means an expected disadvantage. If we view the decision of not-quitting as a relationship-specific investment, then the Responder under-invests because she incurs half of the social cost but capture less than half of the social gain. Given that the Responder quits earlier, the Proposer wants to implement the project earlier to avoid the risk of breakdown. A lower agreement threshold is efficient conditional on having a higher quitting threshold, so the inefficiency comes from the early quitting decision.

The severity of the hold-up problem depends on the relative bargaining power of the two players. When λ is lower than some threshold $\underline{\lambda}$, as in the case of Figure 4a, the bargaining power is very unbalanced. Because the hold-up problem is so strong, the game ends immediately for all initial values. The Responder wants to quit even when the surplus x is higher than sum of the outside options. As a result, the Proposer reacts by making an acceptable offer at all $x > \omega$. The project is implemented if the ex-ante expected surplus is larger than the outside options, and abandoned if it is smaller. If λ is larger than $\underline{\lambda}$, as in Figure X(b), the hold-up problem is less severe and there is a region between $\underline{x}$ and $\overline{x}$ such that players continue to wait. More generally, the agreement threshold increases and the quitting threshold decreases as λ becomes larger.

One can show that, if the only split allowed is an even split, and both players only decide whether to accept that split and whether to quit, then the outcome socially efficient. The uneven split from bargaining causes the weaker player to "under-invest" in the relationship. Thus a higher frequency of counter-offering, which makes the bargaining power more balanced, leads to a more efficient collaboration outcome. As λ approaches ∞, the bargaining power becomes even, and the collaboration outcome approaches the socially efficient outcome.

Corollary 4 (Effect of λ on Social Efficiency)**.** *Ex-ante welfare* $U(x_0) + V(x_0)$ *increases in* λ *(strictly if and only if* $\lambda > \underline{\lambda}$ *and* $\overline{x} > x_0 > \underline{x}$*). As* $\lambda \to \infty$*, the agreement and quitting thresholds approach the socially efficient thresholds, and the utility of each player approaches half of the socially efficient total welfare.*

Corollary 4 says that increasing the frequency of counter-offering does two things. First, it redistributes total utility, just like in the case of static surplus or in the case of irrelevant outside option. The second effect is that it expands the total utility, which is unique to the case of relevant outside option. This implies that a higher frequency of counter-offering may not necessarily be detrimental to the Proposer. Under a higher λ, she may get a smaller share of the pie, but the total size of the pie is larger. If this happens, then both players may prefer to bargain under the higher λ. Proposition 5 proves that this is indeed the case. A higher λ can produce an Pareto-improving outcome, depending on the initial value.

For simpler language, we say that λ_1 *is Pareto Dominated by* λ_2 if the equilibrium outcome under frequency λ_1 is Pareto dominated by the equilibrium outcome under frequency λ_2.

Proposition 5 (Effect of λ on Pareto efficiency)**.**

1. *For an intermediate range of* x_0*, there exists* $\tilde{\lambda}$ *such that any* $\lambda \leq \tilde{\lambda}$ *is Pareto dominated by some* $\lambda' > \tilde{\lambda}$*.*

2. *For all* λ*, there exists a range of* x_0 *such that* λ *is Pareto dominated by some* $\lambda' > \lambda$*.*

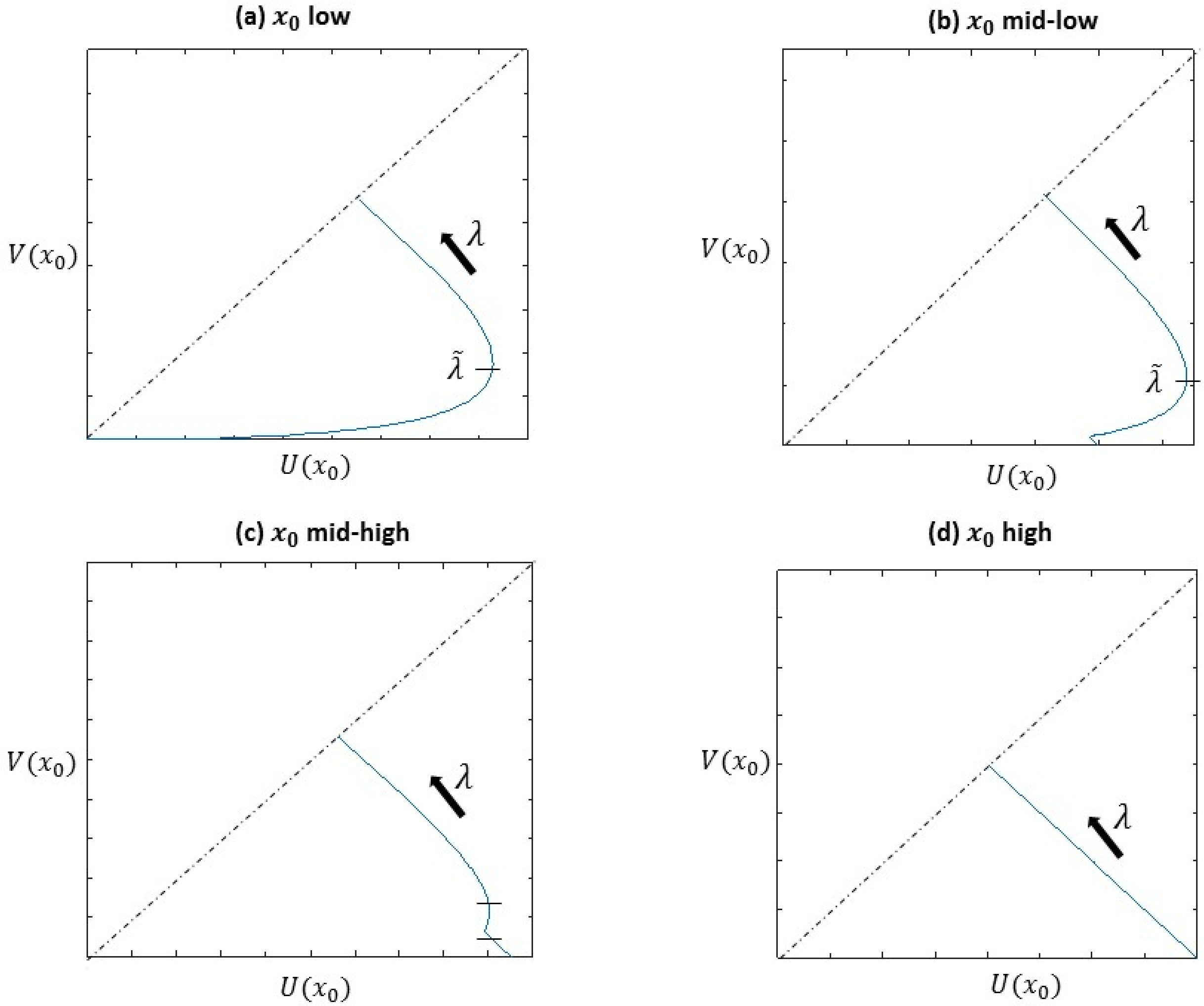
3. *If* $x_0 < \underline{x_s}$ *or* $x_0 > \overline{x_s}$*, then no* λ *is Pareto dominated.*

Proposition 5(1) states that there is an intermediate range of initial values such that all low λ are Pareto dominated. Both players benefit from more frequent counter-offers if the frequency of counter-offers is low. The Proposer benefits from a lower bargaining power, because the balance of power leads to a more efficient timing of collaboration, which outweighs the negative effect of giving up a bigger share to the opponent. Proposition 5(2) states that no choice of λ is immune to Pareto improvement. Regardless of the level of λ, there is always some initial value such that players can mutually benefit from an even higher frequency of counter-offers. Proposition 5(3) states that, if the surplus of the project at the beginning of the negotiation is too low ($x_0 < \underline{x_s}$) or too high ($x_0 > \overline{x_s}$), then the frequency of counter-offers does not matter for Pareto efficiency. The reason is that, in these cases, the outcome is socially optimal regardless of λ, per Corollary 4. The total welfare is not impacted by λ, only the distribution of that welfare.

Figure 5 provides some examples. It outlines the ex-ante utilities of the two players as functions of λ and highlight the Pareto frontiers. The initial value x_0 increases from (a) to (d), and the Pareto dominated region of λ change. In figure 5(a) and 5(b) with low (but still higher than $\underline{x_s}$) initial surplus, all $\lambda < \tilde{\lambda}$ are Pareto dominated, and all $\lambda \geq \tilde{\lambda}$ are on the Pareto frontier for some $\tilde{\lambda}$. In figure 5(c) with a higher initial surplus, some middle levels of λ are dominated, but low λ are on the frontier. In Figure 5(d), the initial surplus is above $\overline{x}$ so that all choices of λ are efficient, since the project is implemented immediately.

Under what procedure then should players bargain? Or alternatively, if we observe players bargain in a certain way, why is such procedure selected? Past literature has been mostly silent on that question. If the bargaining procedure only affects the distribution of welfare, then it is not clear why one extensive-form would be "better" than another. Proposition 5 shows that, when the surplus is stochastic and there is outside option, the choice of bargaining procedure impacts the total welfare. Bargaining with more counter-offers foster a more efficient collaboration on the stopping time, which benefits both players. However, this paper remains agnostic about how the bargaining procedure is determined. I simply assume that a frequency of counter-offers is exogenously set. Future research can explore what happens if the selection is done endogenously by the parties in a negotiation.

Figure 5: Ex-ante utilities as functions of λ for initial values from low to high



5 Conclusion

This paper studies how collaborators of a stochastic project bargain over how the surplus from the project is split, and how the bargaining affects the timing and efficiency of the project implementation. Two symmetric players with time discounting and outside options bargain over a surplus that follows a random walk. The frequency of counter-offers translate to bargaining power, as more frequent counter-offers makes the two parties more balanced. Players takes turn in proposing repeated-offers to the opponent, and the recognition follows a Poisson process. The paper presents the unique symmetric and stationary outcome. When the outside option is non-positive, the project is implemented with socially efficient delay, and the final split of the surplus is more even than under a static surplus. However, when the outside option is strictly positive, the project is implemented too early and the outside option is taken too early compared to the socially efficient thresholds. The inefficiency is caused by the hold-up problem faced by the party responding to the opponent's offers. A higher frequency of counter-offers evens the bargaining power and reduces the severity of the hold-up problem. This increase in social efficiency can outweigh the loss of bargaining power by the proposer. As a result, bargaining with more frequent counter-offers (hence a more balanced procedure) can produce a Pareto improvement. This paper provides theoretical insights on how the bargaining procedure affects the collaboration outcome, and how potential collaborators should or should not bargain.

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A Derivation of Value Functions

If $x \in A$, then the Proposer offers the Responder $p(x)$ and the Responder accepts immediately, so $U(x) = 1 - p(x)$ and $V(x) = p(x)$. If $x \in Q$, then the Responder quits immediately, so $U(x) = V(x) = \omega$. If $x \in \mathcal{R} \setminus (\mathcal{A} \cup \mathcal{Q})$, then the Responder rejects the offer but continues to wait. In this case, the value functions can be written recursively as:

$$\begin{aligned} U(x) &= e^{-rdt}\ \mathbb{E}[\mathbb{1}\{f_{t+dt} = f_t\}U(x+dx) + \mathbb{1}\{f_{t+dt} \neq f_t\}V(x+dx)] + o(dt) \\ V(x) &= e^{-rdt}\ \mathbb{E}[\mathbb{1}\{f_{t+dt} = f_t\}V(x+dx) + \mathbb{1}\{f_{t+dt} \neq f_t\}U(x+dx)] + o(dt) \end{aligned} \tag{4}$$

Since $f_t = 1 + N(t) \bmod 2$, where $N(t)$ follows a Poisson counting process, the probability that the counter-offer event arrives once in dt is $\lambda dt + o(dt)$, and the probability that it arrives more than once is of $o(dt)$. Applying Ito's Lemma to $\mathbb{E}[V(x+dx)]$ and $\mathbb{E}[U(x+dx)]$, one can get:

$$\begin{aligned} U(x) &= e^{-rdt}\ \left\{(1-\lambda dt)\ \left[U(x) + \frac{\sigma^2}{2}U''(x)\right] + \lambda dt\ \left[V(x) + \frac{\sigma^2}{2}V''(x)\right]\right\} + o(dt) \\ &= (1 - rdt)\ \left\{(1-\lambda dt)\ \left[U(x) + \frac{\sigma^2}{2}U''(x)\right] + \lambda dt\ \left[V(x) + \frac{\sigma^2}{2}V''(x)\right]\right\} + o(dt) \\ V(x) &= e^{-rdt}\ \left\{(1-\lambda dt)\ \left[V(x) + \frac{\sigma^2}{2}V''(x)\right] + \lambda dt\ \left[U(x) + \frac{\sigma^2}{2}U''(x)\right]\right\} + o(dt) \\ &= (1 - rdt)\ \left\{(1-\lambda dt)\ \left[V(x) + \frac{\sigma^2}{2}V''(x)\right] + \lambda dt\ \left[U(x) + \frac{\sigma^2}{2}U''(x)\right]\right\} + o(dt) \end{aligned} \tag{5}$$

which after simplification and taking $dt \to 0$ becomes:

$$\begin{aligned} (r+\lambda)U(x) &= \frac{\sigma^2}{2}U''(x) + \lambda V(x) \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{6}$$

Adding and subtracting the two equations produces:

$$\begin{aligned} (U+V)(x) &= \frac{\sigma^2}{2r}(U+V)''(x) \\ (U-V)(x) &= \frac{\sigma^2}{2r+4\lambda}(U-V)''(x) \end{aligned} \tag{7}$$

The solution to these two differential equations is of the form:

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \end{aligned} \tag{8}$$

for some coefficients α_+, α_-, β_+, and β_-.

A.1 Irrelevant Outside Option

Socially Efficient Outcome The social value function, denoted as $W_s(x)$, for $x \leq \overline{x_s}$ is of the form:

$$W_s(x) = C_1 e^{\sqrt{\frac{2r}{\sigma^2}}x} + C_2 e^{-\sqrt{\frac{2r}{\sigma^2}}x} \tag{9}$$

We must have $C_2 = 0$, because the value function has to approach 0 as $x \to -\infty$.

The threshold $\overline{x_s}$ has to satisfy $W_s(\overline{x_s}) = \overline{x_s}$ and $W_s'(\overline{x_s}) = \frac{dx}{dx} = 1$. The second condition is referred to as smooth-pasting and guarantees the optimal timing of the stoppage (See, e.g., Dixit 1993). Solving these two conditions gives the socially efficient threshold

$$\overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$$

and the social value function

$$W_s(x) = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}x-1}$$

.

Equilibrium Outcome Given an agreement threshold $\overline{x}$, player implement the project immediately when the surplus reaches that threshold. Thus for $x \geq \overline{x}$, the sum of their value function equals to the surplus. Also, the Responder must be indifferent between accepting the offer and rejecting the offer in equilibrium, as the Proposer must not offer anything more than necessary. The Responder's value function from rejecting the offer is characterized in equation (6). These imply that the players' value functions for $x \geq \overline{x}$ must satisfy:

$$\begin{aligned} U(x) + V(x) &= x \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{10}$$

Combining the two, one gets:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\sigma^2}{2r+4\lambda}V''(x)$$

which has solution in the form:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_1 e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x \geq \overline{x} \tag{11}$$

for some coefficients γ_1 and γ_2. The Responder can never get more than the full surplus or agree to accept a negative amount, so $0 < V(x) \leq x \ \forall x$; thus $\gamma_1 = 0$, otherwise this is violated for x large enough. So $V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x}$ for some γ_2.

For states below the threshold, the Responder rejects the offer and the project is delayed. Players' value functions must follow the recursive formulations in equations (8). Note that $(U+V)(x)$ captures the social value function, and $(U-V)(x)$ captures the advantage of being the Proposer. As $x \to -\infty$, the social value must approaches zero, which implies

$\beta_+ = 0$. Similarly, as social value approaches zero, the difference between the two players has to approach zero, which implies $\beta_- = 0$. Thus we have a simpler version:

$$(U+V)(x) = \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} \qquad \forall x < \overline{x}$$
$$(U-V)(x) = \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x < \overline{x} \tag{12}$$

Following the proof of Proposition 1, we get $\alpha_+ = \sqrt{\frac{\sigma^2}{2r}} e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}} e^{-1}$, $\alpha_- = \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{-\sqrt{\frac{2r+4\lambda}{2r}}}$, and $\gamma_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{2r}}}$. This pins down the closed-form solutions of the equilibrium value functions:

$$U(x) = \begin{cases} \frac{1}{2}\left(e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} + \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}\right) & \forall x < \sqrt{\frac{\sigma^2}{2r}} \\ \frac{r+\lambda}{r+2\lambda}x - \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)} & \forall x \geq \sqrt{\frac{\sigma^2}{2r}} \end{cases} \tag{13}$$

$$V(x) = \begin{cases} \frac{1}{2}\left(e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} - \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}\right) & \forall x < \sqrt{\frac{\sigma^2}{2r}} \\ \frac{\lambda}{r+2\lambda}x + \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)} & \forall x \geq \sqrt{\frac{\sigma^2}{2r}} \end{cases} \tag{14}$$

where $\overline{x} = \sqrt{\frac{\sigma^2}{2r}}$.

A.2 Relevant Outside Option

Socially Efficient Outcome A social planner with discount rate r and outside option 2ω choose at each moment whether to implement the project, wait, or take the outside option. The social planner implements the project if the surplus reaches above the threshold $\overline{x_s}$, but takes the outside option if the surplus reaches below the threshold $\underline{x_s}$. The social planner's value function has the same form as equations (9):

$$W_s(x) = \gamma_1 e^{\sqrt{\frac{2r}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r}{\sigma^2}}x}$$

The social efficient thresholds have to satisfy:

$$\begin{cases} W_s(\overline{x}) = \overline{x} & W_s(\underline{x}) = 2\omega \\ W_s'(\overline{x}) = 1 & W_s'(\underline{x}) = 0 \end{cases} \tag{15}$$

where the first two are value-matching conditions and the last two are smooth-pasting conditions. Together they imply that the socially efficient thresholds have to be

$$\overline{x_s} = \sqrt{\frac{\sigma^2}{2r} + 4\omega^2} \quad \text{and} \quad \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\left(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\right)$$

The social value function for continuation states $\underline{x_s} < x < \overline{x_s}$ is:

$$W_s(x) = \frac{1}{2}\Big(\overline{x_s} + \sqrt{\frac{\sigma^2}{2r}}\Big)e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} + \frac{1}{2}\Big(\overline{x_s} - \sqrt{\frac{\sigma^2}{2r}}\Big)e^{-\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} \tag{16}$$

Equilibrium Outcome An equilibrium outcome can be described by $\overline{x}$, $\underline{x}$, $U(x)$ and $V(x)$. Given an agreement threshold $\overline{x}$, player implement the project immediately when the surplus reaches that threshold. Thus for $x \geq \overline{x}$, the sum of their value function equals to the surplus. Also, the Responder must be indifferent between accepting the offer and rejecting the offer in equilibrium, as the Proposer must not offer anything more than necessary. The Responder's value function from rejecting the offer is characterized in equation (6). These imply that the players' value functions for $x \geq \overline{x}$ must satisfy:

$$\begin{aligned} U(x) + V(x) &= x \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{17}$$

Combining the two, one gets:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_1 e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x \geq \overline{x} \tag{18}$$

for some coefficients γ_1 and γ_2. The Responder can never get more than the full surplus or agree to accept negative amount, so $0 < V(x) \leq x \ \forall x$; thus $\gamma_1 = 0$, otherwise this is violated for x large enough. So $V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x}$ for some γ_2.

In the case of $\underline{x} \geq \overline{x}$, γ_2 can be solved using the conditions $V(\underline{x}) = \omega$ and $V'(\underline{x}) = 0$. See proof of Proposition 3 for details. The Responder's utility is

$$V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x)} \quad \text{for} \quad x \geq \underline{x} \quad \text{and} \quad V(x) = \omega \quad \text{for} \quad x < \underline{x}$$

The Proposer's utility is

$$U(x) = x - V(x) \text{ for } x \geq 2\omega, \quad \text{and } U(x) = \omega \text{ for } x \leq 2\omega$$

In the case of $\underline{x} < \overline{x}$. There is a region of waiting between $\underline{x}$ and $\overline{x}$. For $\underline{x} < x < \overline{x}$, the value functions follow equations (8):

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \end{aligned} \tag{19}$$

For $x < \underline{x}$, the Responder quits and both players receive $U(x) = V(x) = \omega$.

So the closed-form expressions of equilibrium value functions are:

$$U(x) = \begin{cases} \frac{r+\lambda}{r+2\lambda}x - \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } x \geq \overline{x} \\ \frac{1}{2}\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \frac{1}{2}\beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } \underline{x} < x < \overline{x} \\ \omega & \text{for } x \leq \underline{x} \end{cases} \tag{20}$$

and

$$V(x) = \begin{cases} \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } x \geq \overline{x} \\ \frac{1}{2}\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} - \frac{1}{2}\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} - \frac{1}{2}\beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } \underline{x} < x < \overline{x} \\ \omega & \text{for } x \leq \underline{x} \end{cases} \tag{21}$$

with coefficients

$$\begin{cases} \alpha_+ = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \beta_+ = \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} \\ \gamma_2 = \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} + \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} - \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases} \tag{22}$$

from equations (23) in the proof of Proposition 3, and $\overline{x}$ is the solution to the implicit function.

$$F(\overline{x}, \lambda) = \sqrt{\frac{r}{r+2\lambda}}\Big(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\Big)\Big(\frac{2\omega + \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} + \sqrt{\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2} = 0$$

B Proofs

Proof of Lemma 1 We must have $U(x) + V(x) \geq x \; \forall x$, otherwise the equilibrium does not satisfy sub-game perfection. If the sum of their expected utilities are smaller than the current cake size, then there must be a profitable deviation. To see that, assume there is a state x such that $U(x) + V(x) < x$, then the Responder must accept all offers $p \geq V(x)$. Then the Proposer can propose a $p = V(x)$, and receives immediate payment of $x - V(x)$. Thus $U(x) + V(x) = x$, a contradiction.

Proof of Lemma 2 First, one can show that there must $\exists x$ s.t. $a(p(x)) = 1$ in intervals $(\underline{m}, \infty)$ for all $\underline{m}$. Thus players must reach agreement in some state, and must also reach agreement again for some larger cake size. Suppose they never reach agreement, then $U(x) + V(x) = 0 < x$ for $x > 0$, a contradiction to Lemma 1. Suppose there exists a highest cake size such that players reach agreement, denote that state as $\overline{m}$. Players do not reach agreement for all $x > \overline{m}$. Since players do not quit, then their value functions must be decreasing as x

increases beyond $\overline{m}$. Thus for $x > \overline{m}$, we have $U(x)+V(x) < U(\overline{m})+V(\overline{m}) = \overline{m} < x$, again a contradiction to Lemma 1.

Now define $\overline{x} = inf\{x \mid a(p(x)) = 1\}$. By definition, we establish that $a(p(x)) = 0$ $\forall x < \overline{x}$.

Second, one can show that, if $a(p(x')) = 1$ and $a(p(x'')) = 1$ for some $x' < x''$, then $\nexists$ a open set $Z \subset (x', x'')$ s.t. $a(p(x)) = reject$ $\forall x \in Z$. This means that, players cannot disagree on an open set between two agreement states. Suppose not, then take $x_l = \sup\{x < Z \mid a(p(x)) = accept\}$ as the first state smaller than Z in which players trade, and take $x_r = \inf\{x > Z \mid a(p(x)) = accept\}$ to be the first state bigger than Z in which players trade. Then for any $x \in Z$, players delay until the state reaches x_l or x_r. The probability of reaching x_l first is $\frac{x_r - x}{x_r - x_l}$. Thus without time discounting ($r = 0$), the sum of players value functions must be $U(x)+V(x) = \frac{x_r-x}{x_r-x_l}[U(x_l)+V(x_l)] + \frac{x-x_l}{x_r-x_l}[U(x_r)+V(x_r)] = \frac{x_r-x}{x_r-x_l}x_l + \frac{x-x_l}{x_r-x_l}x_r = x$. If $r > 0$, then we must have $U(x)+V(x) < x$, a contradiction to Lemma 1.

By the utility definition, $A = \{x \mid a(p(x)) = 1\}$ is a closed set. Thus A must be $[\overline{x}, \infty)$.

Proof of Proposition 1 We know that $(U+V)(\overline{x}) = \overline{x}$ as players split a cake of size $\overline{x}$. To prove $\overline{x} = \overline{x_s}$ in equilibrium, we only need to show that $(U+V)'(x) = 1$ hold, because $\overline{x_x}$ is the unique value that satisfies both conditions.

First, we know that $\lim_{x\to\overline{x}^+}(U+V)'(x) = 1$, because $(U+V)(x) = x$ for $x \geq \overline{x}$. If $\lim_{x\to\overline{x}^-}(U+V)'(x) > 1$, then $\exists x < \overline{x}$ such that $(U+V)(x) < x$. Then there's a profitable deviation at that state, a contradiction to Lemma 1. If $\lim_{x\to\overline{x}^-}(U+V)'(x) < 1$, then we must have either $\lim_{x\to\overline{x}^-} U'(x) < \lim_{x\to\overline{x}^+} U'(x)$ or $\lim_{x\to\overline{x}^-} V'(x) < \lim_{x\to\overline{x}^+}$. In either case, the player with the convex kink at $\overline{x}$ can profitably deviate by delaying agreement for an infinitesimal dt. This follows from standard proof of the smooth-pasting condition in optimal stopping problem. (See Dixit 1993 for details) This implies that $\lim_{x\to\overline{x}^-}(U+V)'(x) = 1$, and thus $(U+V)'(\overline{x}) = 1$. Combining with $(U+V)(\overline{x}) = \overline{x}$, we prove that $\overline{x} = \overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$.

Knowing the threshold, now we need to solve for how the cake is split. An accepted offer $p(x)$ must equal to $V(x)$, thus we want to solve for $V(x)$ for $x \geq \overline{x}$. Given $\lim_{x\to\overline{x}^-}(U+V)'(x) = \lim_{x\to\overline{x}^+}(U+V)'(x)$ and that neither player's value function can have a convex kink at $\overline{x}$, we can conclude that:

$$\begin{cases} \lim_{x\to\overline{x}^+} V(x) = \lim_{x\to\overline{x}^-} V(x) \\ \lim_{x\to\overline{x}^+} V'(x) = \lim_{x\to\overline{x}^-} V'(x) \end{cases}$$

Plugging in equations (11), (12), and $\alpha_+ = \sqrt{\frac{\sigma^2}{2r}}e^{-1}$, we get:

$$\begin{cases} \frac{1}{2}\left(\sqrt{\frac{\sigma^2}{2r}}e^{-1}e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}}\right) - \frac{\alpha_-}{2}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} & = \frac{\lambda}{r+2\lambda}\overline{x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \frac{1}{2} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\frac{\alpha_-}{2}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} & = \frac{\lambda}{r+2\lambda} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

Solving the system of equations gives:

$$\begin{cases} \alpha_- &= \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{-\sqrt{\frac{2r+4\lambda}{2r}}} \\ \gamma_2 &= \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{\sqrt{\frac{2r+4\lambda}{2r}}} \end{cases}$$

Plugging in coefficients α_+, α_-, and γ_2 into equations (11) and (12) gives value functions $U(x)$ and $V(x)$. Particularly, for $x \geq \overline{x}$, $V(x) = \frac{\lambda}{r+2\lambda}x + \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{\sqrt{\frac{2r+4\lambda}{2r}}(\overline{x}-x)}$, which shows that the Responder receives strictly more than $\frac{\lambda}{r+2\lambda}x$.

Also $\alpha_- = \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{-\sqrt{\frac{2r+4\lambda}{2r}}} > 0$, so $(U-V)(x) = \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}}$ must also be positive, which concludes the proof.

Proof of Corollary 2 Because $\overline{x} = \overline{x_s}$ from Appendix A.1, the equilibrium outcome must be socially efficient. Taking derivatives of $U(x)$ and $V(x)$ from equations (13) and (14) with respect to λ proves the rest. Particularly, for $x < \overline{x}$, the terms $\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)$, $\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)$, and $e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}$ all decrease in λ, thus $\frac{dU(x)}{d\lambda} < 0$ and $\frac{dV(x)}{d\lambda} > 0$. For $x \geq \overline{x}$,

Proof of Lemma 3 The proof is similar to the proof of Lemma 2.

First, one can show that there must $\exists x$ s.t. $a(p(x)) = 1$ in intervals $(\underline{m}, \infty)$ for all $\underline{m}$. Thus players must reach agreement in some state, and must also reach agreement again for some larger state. Suppose they never reach agreement, then $U(x) + V(x) = 0 < x$ for $x > 0$, a contradiction to Lemma 1. Suppose there exists a highest cake size such that players reach agreement, denote that state as $\overline{m}$. Players do not reach agreement for all $x > \overline{m}$. Then their value functions must be decreasing as x increases beyond $\overline{m}$. Then we have $U(x) + V(x) \leq \overline{m} < x$ for some $x > \overline{m}$, again a contradiction to Lemma 1.

Now define $\overline{x} = inf\{x \mid a(p(x)) = 1\}$. By definition, we establish that $a(p(x)) = 0$ $\forall x < \overline{x}$.

Second, one can show that, if $a(p(x')) = 1$ and $a(p(x'')) = 1$ for some $x' < x''$, then $\nexists$ a open set $Z \subset (x', x'')$ s.t. $a(p(x)) = 0$ $\forall x \in Z$. This means that, players cannot disagree on an open set between two agreement states. Suppose not, then take $x_l = \sup\{x < Z \mid a(p(x)) = 1\}$ as the first state smaller than Z in which players trade, and take $x_r = \inf\{x > Z \mid a(p(x)) = 1\}$ to be the first state bigger than Z in which players trade. Because players trade at x_l, we must have $x_l \geq 2\omega$. Then $Z > 2\omega$. The Responder must not quit in any state in Z. Otherwise, the Proposer can offer ω to the Responder at that state and the Responder accepts, a contradiction to the assumption that players do not trade in Z. Then for any $x \in Z$, players delay until the state reaches x_l or x_r. The probability of reaching x_l first is $\frac{x_r - x}{x_r - x_l}$. Thus without time discounting ($r = 0$), the sum of players value functions must be

$$U(x)+V(x) = \frac{x_r - x}{x_r - x_l}[U(x_l)+V(x_l)] + \frac{x - x_l}{x_r - x_l}[U(x_r)+V(x_r)] = \frac{x_r - x}{x_r - x_l}x_l + \frac{x - x_l}{x_r - x_l}x_r = x$$

If $r > 0$, then we must have $U(x) + V(x) < x$, a contradiction to Lemma 1.

By the utility definition, $A = \{x \mid a(p(x)) = 1\}$ is a closed set. Thus A must be $[\overline{x}, \infty)$.

Next we need to prove that $\overline{x} \leq \overline{x}_s$. Suppose instead $\overline{x} > \overline{x_s}$, then for $x \in (\overline{x_s}, \overline{x})$, we have $U(x) + V(x) \geq x$ by Lemma 1. Also, the total payoff in equilibrium cannot exceed the socially efficient payoff, which is x for $x > \overline{x_s}$. Thus we must have $U(x) + V(x) = x$ for $x \in (\overline{x_s}, \overline{x})$. However, given the form of $U(x)$ and $V(x)$ in equations (8), no parameters can satisfy $U(x) + V(x) = x$ in an open interval, a contradiction.

Proof of Lemma 4 First, one can show that there must $\exists x$ s.t. $q(x) = 1$ in intervals $(-\infty, \overline{m})$ for all $\overline{m}$. Thus the Responder must quit in some state, and must also quit too for some even lower state. Suppose the Responder never quits, then $V(x)$ approaches 0 as $x \to -\infty$. Then the Responder can profitably deviate by quitting. Suppose there exists a lowest state such that the Responder quits. Denote that state as $\underline{m}$, so the Responder does not quit for all $x < \underline{m}$. Then again $V(x)$ approaches 0 as $x \to -\infty$, for which the Responder can deviate by quitting.

Now define $\underline{x} = sup\{x \mid q(x) = 1\}$. By definition, we establish that $q(x) = 0 \ \forall x > \underline{x}$.

Second, one can show that, if $q(x') = 1$ and $q(x'') = 1$ for some $x' < x''$, then $\nexists$ a open set $Z \subset (x', x'')$ s.t. $q(x) = 0 \ \forall x \in Z$. This means that, the Responder cannot choose to continue on any open set between two quitting states. Suppose not, then take $x_l = \sup\{x < Z \mid q(x) = 1\}$ as the first state smaller than Z in which the Responder quits, and take $x_r = \inf\{x > Z \mid q(x) = 1\}$ to be the first state bigger than Z in which the Responder quits. The Responder gets utility of ω at x_l and x_r. If an agreement is reached in this interval, then the Proposer must make an offer such that the Repsonder is indifferent between agreeing or not. Thus for $x_l < x < x_r$, the Responder must gets utility smaller than ω due to discounting. The Responder can profitably deviate by quitting in this interval.

By the utility definition, $Q = \{x \mid q(x)) = 1\}$ is a closed set. Thus Q must be of the form $[-\infty, \underline{x})$.

Third, we need to prove that $\underline{x} \geq \underline{x}_s$ from Section 4.1 and Appendix A.2. Suppose instead $\underline{x} < \underline{x_s}$. For states $x < \underline{x}$, the socially efficient outcome is quitting, and the equilibrium outcome cannot have a higher payoff than the socially efficient payoff. Thus $U(x) + V(x) \leq 2\omega$. Because the Responder does not quit in the region $(\underline{x}, \underline{x_s}$, we must have $V(x) \geq \omega$ in this region. This implies $U(x) \leq \omega$ in $(\underline{x}, \underline{x_s}$. By the form of $U(x)$ and $V(x)$ from equations (2), they cannot be flat functions. Then there must exists some $x' > \underline{x}$ such that $U(x') < V(x')$ and $U'(x') < V'(x')$. This implies that $(U - V)(x') < 0$ and $(U - V)'(x') < 0$.

Again by equations (2), if $(U - V)(x') < 0$ then $(U - V)''(x') < 0$, which implies that $(U - V)'(x) < 0$ and $(U - V)(x) < 0$ for $\underline{x} < x < \overline{x}$. Thus we must have $U'_-(\overline{x}) < V'_-(\overline{x})$ and $U(\overline{x}) < V(\overline{x})$. Because $(U + V)(x) = x$ for $x \geq \overline{x}$, we must have $(U + V)'_-(\overline{x}) \leq 1$, otherwise there exists some $x < \overline{x}$ such that $(U + V)(x) < x$, a contradiction to Lemma 1. Thus $U'_-(\overline{x}) < \frac{1}{2}$.

Because $U(\overline{x}) < V(\overline{x})$, γ_2 from equation (18) must be positive. Then by equations (17) and (18), we have $U'_+(\overline{x}) > V'_+(\overline{x})$. Because $(U + V)(x) = x$ for $x \geq \overline{x}$, we can conclude that $U'_+(x) > \frac{1}{2}$.

Because $U'_-(\overline{x}) < \frac{1}{2}$ and $U'_+(x) > \frac{1}{2}$, there is a convex kink on $U(x)$ at $\overline{x}$. Then the Proposer can profitably deviate by delaying the trade for a small dt. See Dixit (1993), pg.xx for details. Thus we cannot have $\underline{x} < \underline{x_s}$ in equilibrium, concluding the proof.

Proof of Proposition 3 We solve for the equilibrium outcome in two cases.

Case 1: $\underline{x} \geq \overline{x}$:

The first case is $\underline{x} > \overline{x}$. In this case, the game always end immediately. So the players must trade for $x_0 > 2\omega$. Thus $\overline{x} = 2\omega$. On equilibrium path, players trade immediately for $x_0 \geq 2\omega$, and the Responder quit immediately for $x_0 < 2\omega$. We know that $V(x) = \omega$ for $x \leq \underline{x}$ is the Responder's equilibrium payoff when she quits. Then we have $V'_-(\underline{x}) = 0$. In order for the quitting threshold to be optimal, we need to have $V'_+(\underline{x}) = 0$, otherwise the Responder can profitably deviate by delaying quitting for time dt. Thus we have

$$V(\underline{x}) = \omega \quad \textbf{and} \quad V'(\underline{x}) = 0$$

The form of $V(x)$ is given in equation (18) in Appendix A.2. Plugging in equation (18) with $\gamma_1 = 0$, we get:

$$\begin{cases} \frac{\lambda}{r+2\lambda}\underline{x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = \omega \\ \frac{\lambda}{r+2\lambda} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = 0 \end{cases}$$

Solving the two equations, we find that the solution exists if and only if $\lambda \leq \underline{\lambda} = \frac{4r^2\omega^2+\sqrt{16r^4\omega^4+8\sigma^2r^3\omega^2}}{2\sigma^2}$. The solutions are:

$$\begin{cases} \underline{x} = \frac{r+2\lambda}{\lambda}\omega - \sqrt{\frac{\sigma^2}{2r+4\lambda}} \\ \gamma_2 = \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} \end{cases}$$

The Responder's utility is $V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x)}$ for $x \geq \underline{x}$ and $V(x) = \omega$ for $x < \underline{x}$. The Proposer's utility is $U(x) = x - V(x)$ for $x \geq \underline{x}$, $U(x) = x - \omega$ for $2\omega < x < \underline{x}$, and $U(x) = \omega$ for $x \leq 2\omega$.

Case 2: $\underline{x} < \overline{x}$:

An equilibrium outcome has to satisfy the following four value-matching conditions:

$$\begin{cases} (U+V)(\underline{x}) = 2\omega \\ (U-V)(\underline{x}) = 0 \\ (U+V)(\overline{x}) = \overline{x} \\ (U-V)(\overline{x}) = \overline{x} - 2V(\overline{x}) = \frac{r}{r+2\lambda}\underline{x} - 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

where $V(\overline{x})$ comes from equation (18). An equilibrium outcome also has to satisfy the following three conditions:

1. $V'(\underline{x}) = 0$ or $(U+V)'(\underline{x}) = (U-V)'(\underline{x})$ for optimality of the quitting threshold.

2. $(U+V)'(\overline{x}) = 1$. We know that $(U+V)'_+(\overline{x}) = 1$ since $(U+V)(x) = x$ for $x > \overline{x}$. If $(U+V)'_-(\overline{x}) > 1$, then one of the player has a convex kink on his/her value function at $\overline{x}$, and can profitably deviate and delaying trade for dt. If $(U+V)'_-(\overline{x}) < 1$, then $(U+V)(x) < x$ for some $x < \overline{x}$, a contradiction to Lemma 1.

3. $(U-V)'_-(\overline{x}) = (U-V)'_+(\overline{x})$. We know that $(U+V)'_-(\overline{x}) = (U+V)'_+(\overline{x}) = 1$. Also, there cannot be convex kink for either $U(x)$ or $V(x)$ at $\overline{x}$, otherwise a profitable deviation exists. Thus $U'_-(\overline{x}) = U'_+(\overline{x})$ and $V'_-(\overline{x}) = V'_+(\overline{x})$.

Using the forms of $(U+V)(x)$ and $(U-V)(x)$ from equations (8) for the region $\underline{x} < x < \overline{x}$, the above seven conditions constitute the following system of equations:

$$\begin{cases} \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}} = 2\omega \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = 0 \\ \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \overline{x} \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} = \frac{r}{r+2\lambda}\overline{x} - 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \sqrt{\frac{2r}{\sigma^2}}(\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}}) = \sqrt{\frac{2r+4\lambda}{\sigma^2}}(\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}}) \\ \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} - \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \sqrt{\frac{\sigma^2}{2r}} \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} - \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} = \frac{r}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} + 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

Solving the system of equations, we get the following implicit equation of $\underline{x}$ and λ:

$$F(\overline{x},\lambda) = \sqrt{\frac{r}{r+2\lambda}}\Big(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\Big)\Big(\frac{2\omega + \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} + \sqrt{\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2} = 0$$

By Lemma 3, $2\omega \le \overline{x} \le \overline{x_s}$. Here I use a Lemma that I prove in the Online Appendix.

Lemma 5. *The equation* $F(\overline{x},\lambda)$ *has a unique solution* $\overline{x}(\lambda)$ *in the range of* $2\omega \le \overline{x} \le \overline{x_s} = \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}$. *The solution* $\overline{x}(\lambda)$ *is increasing in* λ *for* $\lambda \ge \underline{\lambda}$, $\overline{x}(\underline{\lambda}) = 2\omega$, *and* $\overline{x} \to \overline{x_s}$ *as* $\lambda \to \infty$.

By Lemma 5, the equilibrium is unique. If $\lambda \le \underline{\lambda}$, then the equilibrium outcome is in case 1. In equilibrium, $\underline{x} \ge \overline{x}$, and the game ends immediately. If $\lambda > \underline{\lambda}$, then the equilibrium outcome is in case 2. There exists a unique $\overline{x}$ as a function of λ in the range of $(2\omega, \overline{x_s})$. We can solve the rest of the parameters as functions of λ and $\overline{x}$. The solutions are:

$$\begin{cases} \alpha_+ = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \beta_+ = \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}}\log\frac{2\omega - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} - \sqrt{\frac{\sigma^2}{2r}}} \\ \alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} \\ \gamma_2 = \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} + \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} - \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases} \quad (23)$$

To prove Proposition 3(1): Note that case 1 only exists for $\lambda \le \underline{\lambda}$, and case 2 only exists for $\lambda > \underline{\lambda}$. So for each λ, there is a unique equilibrium outcome. From Lemma 5 we get $\frac{d\overline{x}}{d\lambda} > 0$ and $\overline{x} \to \overline{x_s}$ as $\lambda \to \infty$. Thus $\overline{x} < \overline{x_s}$ for all λ. Using equations (23) we can then conclude that $\underline{x} > \underline{x_s}$ if $\overline{x} < \overline{x_s}$.

Next we need to prove that $(U-V)(x)\geq 0$ for all x. Suppose $\lambda \leq \underline{\lambda}$. Then $(U-V)(x)=0$ for $x\leq 2\omega$, $(U-V)(x)=x-2\omega$ for $x\in(\omega,\underline{x})$, and $(U-V)(x)=\frac{r}{r+2\lambda}x-\frac{2\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x}$ for $x\geq \underline{x}$. Then $\frac{d(U-V)}{dx}\geq 0$, so $U(x)\geq V(x)$ for all x.

Suppose $\lambda > \underline{\lambda}$. Then $(U-V)(x)=0$ for $x\leq \underline{x}$. For $x\in(\underline{x},\overline{x})$,

$$\begin{aligned}
(U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x}+\beta_-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x\\
&= \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}-\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-x-\overline{x})}\\
&= \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})\left[e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}-e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-x-\overline{x})}\right]\\
&\geq 0
\end{aligned}$$

because $x-\overline{x}>2\underline{x}-x-\overline{x}$. For $x\geq\overline{x}$,

$$\begin{aligned}
(U-V)(x) &= \frac{r}{r+2\lambda}x-2\gamma_2 e^{\sqrt{-\frac{2r+4\lambda}{\sigma^2}}x}\\
&= \frac{r}{r+2\lambda}x-\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-x-\overline{x})}-\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}-\sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)}\\
&\geq \frac{r}{r+2\lambda}x-\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})-\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x}-\sqrt{\frac{\sigma^2}{2r+4\lambda}})\\
&= 0
\end{aligned}$$

Thus the Proposer has a weakly higher utility in all states x and for all λ.

For Proposition 3(2): The only thing left to prove is that $\frac{d\underline{x}}{d\lambda}<0$ for $\lambda>\underline{\lambda}$. Because $\frac{d\overline{x}}{d\lambda}>0$, we just need to prove $\frac{d\underline{x}}{d\overline{x}}<0$. From equations (23), one can derive that $e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-\underline{x})}=\frac{2\omega-\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}{\overline{x}-\sqrt{\frac{\sigma^2}{2r}}}$. Then:

$$\frac{de^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-\underline{x})}}{d\overline{x}}=\sqrt{\frac{2r}{\sigma^2}}e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-\underline{x})}(1-\frac{d\overline{x}}{d\underline{x}})=\frac{(\overline{x}-\sqrt{\frac{\sigma^2}{2r}})(\frac{x}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}})-(2\omega-\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}})}{(\overline{x}-\sqrt{\frac{\sigma^2}{2r}})^2}$$

Thus

$$\begin{aligned}
(1-\frac{d\overline{x}}{d\underline{x}}) &= \frac{\sqrt{\frac{\sigma^2}{2r}}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}\frac{\overline{x}}{2\omega-\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}-\frac{\sqrt{\frac{2r}{\sigma^2}}}{\overline{x}-\sqrt{\frac{2r}{\sigma^2}}}\\
&\geq \frac{\overline{x}}{\overline{x}-\sqrt{\frac{2r}{\sigma^2}}}-\frac{\sqrt{\frac{\sigma^2}{2r}}}{\overline{x}-\sqrt{\frac{\sigma^2}{2r}}}=1
\end{aligned}$$

using the fact that $\overline{x} \geq 2\omega$. This implies that $\frac{d\underline{x}}{d\overline{x}} < 0$, concluding the proof. The quitting threshold decreases in λ. If $\overline{x} = \overline{x_s}$, one can check that $\underline{x} = \underline{x_s}$. So $\underline{x}$ approaches $\underline{x_s}$ as $\lambda \to \infty$.

Proof of Corollary 4 If $\lambda \leq \underline{\lambda}$, then game ends immediately, so $(U + V)(x) = \max\{x_0, 2\omega\}$. Ex-ante welfare is not affected by λ.

Suppose $\lambda > \underline{\lambda}$. Plugging coefficients from equations (23) into $(U+V)(x)$ from equations (8), we get:

$$(U + V)(x) = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} + \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)}$$

for $\underline{x} < x < \overline{x}$. The derivative with respect to λ is:

$$\frac{d(U + V)(x)}{d\lambda} = \frac{d(U + V)(x)}{d\overline{x}}\frac{d\overline{x}}{d\lambda} = \frac{1}{2}\sqrt{\frac{2r}{\sigma^2}}\overline{x}(e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)} - e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)})\frac{d\overline{x}}{d\lambda}$$

Because $\frac{1}{2}\sqrt{\frac{2r}{\sigma^2}}\overline{x}(e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)} - e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)}) > 0$ and $\frac{d\overline{x}}{d\lambda} > 0$ from Lemma 5, we can conclude that $\frac{d(U+V)(x)}{d\lambda} > 0$. Thus if $x_0 \in (\underline{x}, \overline{x})$, then the ex-ante welfare is strictly increasing in λ for $\lambda < \underline{\lambda}$. If $x_0 \notin (\underline{x}, \overline{x})$, then the game ends immediately and the ex-ante welfare is not affected by λ.

As $\lambda \to \infty$, $\overline{x} \to \overline{x_s}$ by Lemma 5. As $\overline{x} \to \overline{x_s}$, we have $\underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}} \log \frac{2\omega - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} - \sqrt{\frac{\sigma^2}{2r}}} \to \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\left(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\right)$. Also $(U + V)(x) \to (U + V)_s(x)$ from equation (16) in Appendix A.2. So the total welfare approaches the socially efficient welfare. Also, both $\alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}}$ and $\beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})}$ approach 0 as $\lambda \to \infty$, so $(U - V)(x) \to 0$. Thus $(U + V)(x_0) = (U + V)_s(x_0)$ and $(U - V)(x_0) = 0$ in the limit, so each player approaches half of the socially efficient total welfare.

Proof of Proposition 5 First we prove Proposition 5(1). Let $I = (\underline{x_s}, 2\omega)$, and $\tilde{\lambda} = \underline{\lambda}$. If $\lambda \leq \underline{x}$, then $\underline{x} \geq \overline{x}$ by Proposition 3, so the game ends immediately. If $x_0 \in I$, then the Responder quits at time 0 and both players get outside option of ω. Again by Proposition 3, as $\lambda \to \infty$, $\underline{x} \to \underline{x_s}$ and $(U + V)(x_0) \to (U + V)_s(x_0)$. So there exists a $\lambda' > \underline{\lambda}$ such that $\underline{x}(\lambda') < x_0$ and $(U + V)(x_0) > 2\omega$, because $(U + V)_s(x_0) > 2\omega$. So the total ex-ante welfare is higher under λ'. The Responder can never be worse off than ω, and the Proposer has weakly higher utility than the Responder. Thus λ' Pareto dominates λ.

For Proposition 5(2): For any λ, $\underline{x}(\lambda) > \underline{x_s}$. Take any x_0 in the interval $(\underline{x_s}, \underline{x})$. For such x_0, the ex-ante utilities for both players under λ is ω. Then by Proposition 3, as $\lambda \to \infty$, $\underline{x} \to \underline{x_s}$ and $(U + V)(x_0) \to (U + V)_s(x_0)$. So there exists a $\lambda' > \lambda$ such that $\underline{x}(\lambda') < x_0$ and $(U + V)(x_0) > 2\omega$. This must be a Pareto improvement, similarly to the argument above.

For Proposition 5(3): If $x_0 < \underline{x_s}$ or $x_0 > \overline{x_s}$, then it is socially optimal to stop immediately. The equilibrium outcome is socially optimal regardless of λ, so total welfare does change. So no λ can Pareto dominates another.

C Allowing Proposer to Quit

TBD

D Online Appendix

Lemma 5

$$F(\overline{x},\lambda)=\sqrt{\frac{r}{r+2\lambda}}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})(\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}})^{\sqrt{\frac{r+2\lambda}{r}}}-\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}=0$$

has a unique solution $\overline{x}(\lambda)$ in the range of $2\omega\le\overline{x}\le\overline{x_s}=\sqrt{\frac{\sigma^2}{2r}+4\omega^2}$. The solution $\overline{x}(\lambda)$ is increasing in λ for $\lambda\ge\underline{\lambda}$, $\overline{x}(\underline{\lambda})=2\omega$, and $\overline{x}\to\overline{x_s}$ as $\lambda\to\infty$.

Proof. First, $\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}\le 1$ because $\overline{x}\ge 2\omega$ by Lemma 3. Thus $\frac{\partial F}{\partial\lambda}<0$ because every term decreases in λ.

By implicit function theorem, to prove $\frac{\partial\overline{x}}{\partial\lambda}>0$, we need to prove that $\frac{\partial F}{\partial\overline{x}}>0$ for $2\omega\le\overline{x}\le\overline{x_s}$. We prove by the following steps. We show that $\frac{\partial F}{\partial\overline{x}}=M(\overline{x})*E(\overline{x})$ for some M and E. The term $M(\overline{x})$ is always positive. The term $E(\overline{x})$ is positive at $\overline{x}=2\omega$, and $\frac{\partial E}{\partial\overline{x}}>0$ for $\overline{x}>2\omega$, thus concluding that $\frac{\partial F}{\partial\overline{x}}>0$.

Let $M=\left(\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\frac{\sigma^2}{2r}}\right)^{\sqrt{\frac{r+2\lambda}{r}}}$. Then $M>0$ for $\overline{x}<\overline{x_s}=\sqrt{4\omega^2+\frac{\sigma^2}{2r}}$, then:

$$\begin{aligned}\frac{\partial F}{\partial\overline{x}}&=\frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}-\frac{\overline{x}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}\frac{1}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}M-\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}M+M\\&=M*\Big[\frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}(\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}})^{\sqrt{\frac{r+2\lambda}{r}}}\\&\quad-\frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}-\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}+1\Big]\\&=M*E(\overline{x})\end{aligned}\tag{24}$$

At $\overline{x}=2\omega$, $E(\overline{x})=\left(\frac{2\omega}{\sqrt{\frac{\sigma^2}{2r}+1}}\right)\left(\frac{2\omega+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{\frac{\sigma^2}{2r}}}\right)>0$. We can write $E(\overline{x})$ as $E=\frac{\overline{x}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}*B-$

$\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}} + 1$, where

$$\begin{aligned} B &= \Big(\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} \\ &> \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} \\ &= \frac{\sqrt{\frac{\sigma^2}{2r}}-\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} > 0 \end{aligned} \tag{25}$$

with $\frac{dB}{d\overline{x}} > 0$. Then taking derivative of E with respect to $\overline{x}$, we get:

$$\frac{dE}{d\overline{x}} = \frac{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}+\overline{x}^2/\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}^2} * B + \frac{\overline{x}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}\frac{dB}{d\overline{x}} - \frac{\sqrt{\frac{\sigma^2}{2r}}-\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{(\overline{x}+\sqrt{\frac{\sigma^2}{2r}})^2} \tag{26}$$

Then $\frac{dE}{d\overline{x}}$ is positive because each of the terms is positive for $\overline{x} \in (2\omega, \overline{x_s})$. Because $E(\overline{x} = 2\omega) > 0$, this implies $E(x) > 0$ for all $\overline{x} \geq 2\omega$. Thus $\frac{dF}{d\overline{x}} = M * E > 0$ for $\overline{x} \in\in (2\omega, \overline{x_s})$. Combining with the fact that $\frac{dF}{d\lambda} < 0$, by implicit function theorem, there exists a function $\overline{x}(\lambda)$ in that range and $\frac{d\overline{x}}{d\lambda} > 0$.

□