

FOLLOWING THE CUSTOMERS: DYNAMIC COMPETITIVE
REPOSITIONING*

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ABSTRACT

We consider dynamic repositioning when competing firms try to follow the evolution of consumer preferences, while taking into account the competitive interaction, both in terms of static market competition, and the dynamic effects of different firm positionings. We fully characterize the dynamic market equilibrium, which includes the timing of the firms' repositionings depending on consumer preferences. As consumer preferences evolve away from where both firms are located, one firm first moves to follow consumer preferences, with the second firm only moving if the consumer preferences continue evolving away from that firm. The model predicts rich market dynamics, where firms stay for some period in different positionings if consumer preferences are in a relatively middle ground, or where a firm repositions to follow consumer preferences, but then repositions back to the original position, if consumer preferences return. We find that, when the variability of the consumer preferences or the discount rate is greater, or when the importance of the repositioning attribute is smaller, firms are less likely to follow consumer preferences and are more heterogeneous in their responses. We also find that competing firms reposition less frequently than what is socially optimal, and than what collusion would imply.

1. INTRODUCTION

Consumer preferences evolve over time, and firms have to keep adjusting their product offerings to follow consumer preferences. Those product adjustments, which end up repositioning the products, are costly, and firms have to decide when to incur those costs depending on how far the consumer preferences are from the current offering, as well as their competitors' own positioning and potential decision to reposition. For example, in the automobile market, when consumer preferences evolved for a greater appreciation for sport utility vehicles or for electric vehicles, the automobile manufacturers had to decide to make the move to this type of vehicle, depending on both consumer preferences and the potential strategic investments of their competitors.

We consider a dynamic model under competition, where firms have to decide when to invest in repositioning their products given the evolution of preferences, and the repositioning decisions of the competitors. That is, firms follow consumer preferences while keeping in mind the market interaction with competitors.

The model predicts rich market dynamics, where firms stay for some period in different positionings if consumer preferences are in a relatively middle ground, or where a firm repositions to follow consumer preferences, but then repositions back to the original position, if consumer preferences return back to the original position. For example, when Coca-Cola perceived that consumers had changed their preferences towards sweeter and smoother sodas, it introduced New Coke; once it learned that the consumer preferences had a strong preference for the traditional taste, Coca-Cola went back to its original positioning (Classic Coke).

We find that firms are less likely to follow consumer preferences if there is greater variability over time in consumer preferences. With greater variability of consumer preferences, firms wait longer to reposition, as it is more likely that the consumer preferences will return to where the firms are positioned. Similarly, if firms' discount rate is higher, firms are less willing to reposition because the present value of the gains from repositioning are now lower.

We also investigate the likelihood of one firm following a competitor in repositioning when consumer preferences change. This can also be seen as related to the likelihood of the firms having different positions in the market. We find that the greater the variability of consumer preferences, the less likely is a firm to follow the competitor in repositioning, which yields a greater likelihood of firms having a different positioning in the market. Similarly, if the discount rate is greater, a firm is also less likely to follow the competitor's repositioning, which

leads to a greater likelihood of firms being positioned differently in the market. Additionally, firms are less likely to be differentiated when the repositioning attribute is more important, as competition intensifies.

We also compare the firm repositioning in competition with what would be socially optimal. We find that competing firms reposition less frequently than what is socially optimal, as the competing firms only account for the private benefits of repositioning. We also find that competing firms reposition less frequently than collusive firms, if we allow for collusion on prices.

There has been substantial research on static positioning in markets (e.g., Hotelling 1929, Hauser 1988, Moorthy 1988, Sayman et al. 2002, Lauga and Ofek 2011), with particular focus on the competitive interaction. There is also work on the effects of the resources of the firms on their strategic positioning (e.g., Wernerfelt 1989). With dynamics, there is work on investments in R&D that generate with a certain probability success in the repositioning of the product (e.g., Harris and Vickers 1987, Ofek and Sarvary 2003). In contrast, this paper allows for the decision to reposition to have immediate effects, and therefore the timing of when to reposition a product in competition becomes the crucial decision. Furthermore, here the consumer preferences move in non-predictable ways, so it becomes important to understand when firms follow consumer preferences. Another related paper is Budd, Harris, and Vickers (1993), where firms compete dynamically to gain market share, where market share evolves stochastically and continuously, depending on the firms' efforts. Here market share can evolve discontinuously because of repositioning decisions by firms.¹

A similar decision to the one considered here is whether and when to adopt new technologies, which is considered in a two-state version in Villas-Boas (1992). This paper considers a richer, uncertain environment, where the decision on when to reposition is investigated in greater depth. Another related stream of work considers richer environments of dynamic competition in R&D among firms that is presented for empirical work and which can be solved with numerical methods (e.g., Ericson and Pakes 1995, and, in particular with dynamic repositioning, Sweeting 2013, Jeziorski 2014). In relation to that work, this paper presents sharper analysis of when to reposition, and how that decision depends on the degree of uncertainty in the market and on the discount rate.²

¹See also Cabral and Riordan (1994), where firms compete on learning-by-doing, with actions in each period affecting the future costs.

²Also related to this paper, but here with competition, is the literature on portfolio choice with transaction costs, where an investor only adjusts the portfolio once in a while because of transaction costs and the portfolio evolves stochastically (e.g., Magill and Constantinidis 1976), and the literature on (S,s) economies

The remainder of the paper is organized as follows. The next section presents the model of competition and consumer preferences. Section 3 presents the competitive market equilibrium, and Section 4 considers the expected duration when firms are differentiated or positioned in the same location. Section 5 studies what would happen if there were collusion, and Section 6 presents the social welfare problem and compares it with the competitive equilibrium and collusive outcome. Section 7 concludes.

2. THE MODEL

Consider a market with two competing firms selling to a unit mass of consumers. Each consumer buys from one of the two firms at each moment in time and time is continuous. Consumer preferences are characterized by $(x, z) \in [0, 1]^2$, where all consumers have the same x and are uniformly distributed on the dimension z .

The location of x evolves over time, and we denote by x_t the location of the consumer preferences at time t . The position x_t evolves over time as a Brownian motion with variance σ^2 , and reflecting boundaries at 0 and 1.

Firms can reposition on the dimension x but they cannot reposition on the dimension z . On dimension z , firms are positioned on the opposite extremes of the segment $[0, 1]$. On dimension x , firms can be positioned at one, and only one, of the two ends of a segment $[0, 1]$. At any moment, a firm can choose to reposition to the other side of the segment. Note that, depending on the repositioning decisions, firms on dimension x can have the same positioning or different positionings. The repositioning is instantaneous but with a cost of K . Firms are symmetric with a discount rate r . We assume without loss of generality that Firm 1 is positioned at location 0. If Firm 1 repositions, we relabel its new position as 0 and the opposite side as 1. Figure 1 illustrates how consumer preferences could evolve over time and the possible repositioning of the two firms.

A consumer gets utility of $v - \delta x_t - z - p_1$ when she purchases from Firm 1, where p_1 is the price charged by Firm 1. We assume v is high so that the market is fully covered. Dimension z represents the idea that firms never have exactly the same positioning; firms still have positive profits if they are positioned at the same point on dimension x . The assumed uniform distribution of z allows us to obtain sharper results. The parameter δ measures the

from inventory problems (e.g., Scarf 1959, Sheshinski and Weiss 1983). See also Villas-Boas (2018) for the dynamic repositioning problem in a monopoly setting. For a recent paper of dynamic quantity competition with a diffusion process (with incomplete information) see Bonatti, Cisternas, and Toikka (2017).

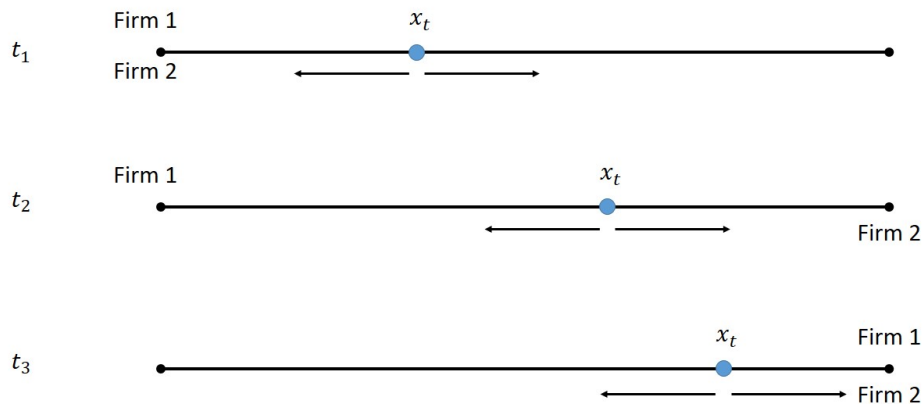


Figure 1: Possible positions of consumers and firms over time

importance of dimension x relative to dimension z in consumer preferences. If δ is very high, consumers care mostly about the dimension on which firms can reposition. If δ is very low, consumers do not care too much about the dimension on which the firms can reposition.

Firms engage in price competition. Let π_s denote a firm's instantaneous profit when firms are located on the same side. Let $\pi_d(x_t)$ denote a firm's instantaneous profit when firms are located on different sides, where x_t is the distance between the firm and the consumers' ideal location. The instantaneous profits are $\pi_s = \frac{1}{2}$ and $\pi_d = \frac{1}{2}p_1^2$ where $p_1 = 1 + \frac{\delta(1-2x)}{3}$.

We look for a symmetric Markov perfect equilibrium of this game. That is, strategies depend on the payoff relevant state variables. In this case, the payoff relevant state variables are the current positioning of the firms, and the distance x_t of the consumer preferences in dimension x to the positioning of Firm 1. We allow firms to play behavioral strategies, that is, mixed strategies at each moment in time. A behavioral strategy at a given state x_t is represented by a function $\mu(x_t)$, which gives the hazard rate at which a firm repositions.

Let us conclude this Section by briefly discussing some of the model assumptions. The model considers two specific positionings for the product. This simplifies the model greatly, while still capturing the effects of the firms following consumer preferences and of the competitive interaction. A more general formulation could consider more (or a continuum of) positioning possibilities, and consumer preferences that are not limited to evolve within a finite segment. The model also restricts attention to the two-firm case, and it would be interesting also to investigate what happens with more than two firms. Finally, the model

restricts attention to the case of the market being fully covered. This allows us to focus on the competitive interaction effects, but a more general model could allow for less-than-full market coverage.

3. MARKET EQUILIBRIUM

3.1. Form of the Equilibrium

We now study the symmetric dynamic market equilibrium of this market. We first discuss the potential form of the equilibrium, and then construct the market equilibrium.

3.1.1. Differentiated Firms

If firms are positioned on different sides, their equilibrium strategy should feature a threshold x^* such that a firm repositions when its distance to consumers, x_t , hits x^* . The opposing firm repositions when x_t hits $1 - x^*$. That is, if firms are positioned on different sides of dimension x , they continue to be so differentiated as long as $x_t \in [1 - x^*, x^*]$, where we expect x^* to be greater than $1/2$.

3.1.2. Non-Differentiated Firms

Now suppose that firms are positioned on the same side. Consider first that firms use pure strategies. Suppose that the strategy of one firm is to reposition for some $x_t < x^*$. The competitor has two potentially profitable deviations. It can let the opponent move and wait until x^* to reposition itself. By definition of x^* , this is always preferred to moving together at x_t . Alternatively, it can preempt the opponent's repositioning by some ε , which forces the opponent to wait until x^* to reposition. One of these two strategies must be strictly preferred to relocating together at x_t . Then the only possible pure strategy equilibrium would be an asymmetric equilibrium where one firm moves at some $x_t < x^*$, and the other firm follows at x^* . We present such a possible equilibrium in the Appendix.

Given our focus on the symmetric equilibrium, we look for behavioral strategy equilibria in which firms mix their repositioning decisions at each node. Firms on the same side reposition with some hazard rate $\mu(x)$. The hazard rate can only be positive for a range of x between $1/2$ and x^* , because $\pi_d(1 - x) > \pi_d(x)$ if and only if $x > 1/2$. Firms have no

incentive to move if they end up farther away from consumers after moving. In the remainder of the paper, we study this equilibrium characterized by x^* and $\mu(x)$.

To complete the equilibrium characterization, we need to consider what happens when firms are positioned on the same side with $x_t \geq x^*$. From previous analysis, we know that in equilibrium firms cannot be located on different sides with $x_t \geq x^*$ because the firm farther away from consumers should relocate. Suppose one firm moves at x_t so they become differentiated, then the other firm strictly prefers to follow immediately because $x_t \geq x^*$. They both incur repositioning cost K but do not gain in flow profit, because firms are still positioned on the same side. Repositioning only increases total industry revenue if it results in differentiation. Thus if one firm does not reposition at x , then the other firm also does not have incentive to do so. In this range, repositioning becomes a coordination problem. For example, one equilibrium could be that firms never reposition for $x_t \geq x^*$. Another could be that they reposition for all $x_t \geq x^*$. The first equilibrium gives a higher payoff to both firms by reducing total repositioning costs. In fact, the equilibrium where firms never reposition for $x_t \geq x^*$ when they are on the same side is Pareto dominant among symmetric equilibria, because it eliminates repositioning that does not lead to differentiation. In this paper, we focus on this Pareto dominant equilibrium. As we will see below, strategy for $x_t \geq x^*$ is payoff irrelevant after some possible initial repositioning. Because the equilibrium involves one firm repositioning with probability one for some $x_t < x^*$, the case where $x_t > x^*$ and both firms are on the same side is never on the equilibrium path, except, potentially, for some initial period of time if the market starts with both firms on the same side and $x_0 > x^*$.

To summarize, a market equilibrium is going to be characterized by an x^* and a $\mu(x)$. When firms are positioned on different sides, a firm repositions if its distance to the consumers reaches x^* . When firms are positioned on the same side, a firm repositions with hazard rate $\mu(x)$. Figure 2 illustrates the market equilibrium depending on the positioning of the firms and the location of the consumer preferences.

3.2. Value Functions and Boundary Conditions

We now construct the equilibrium using the objective functions of the firms at the possible different situations.

When firms are at different locations, and $x \in [1 - x^*, x^*]$, a firm's value function as a

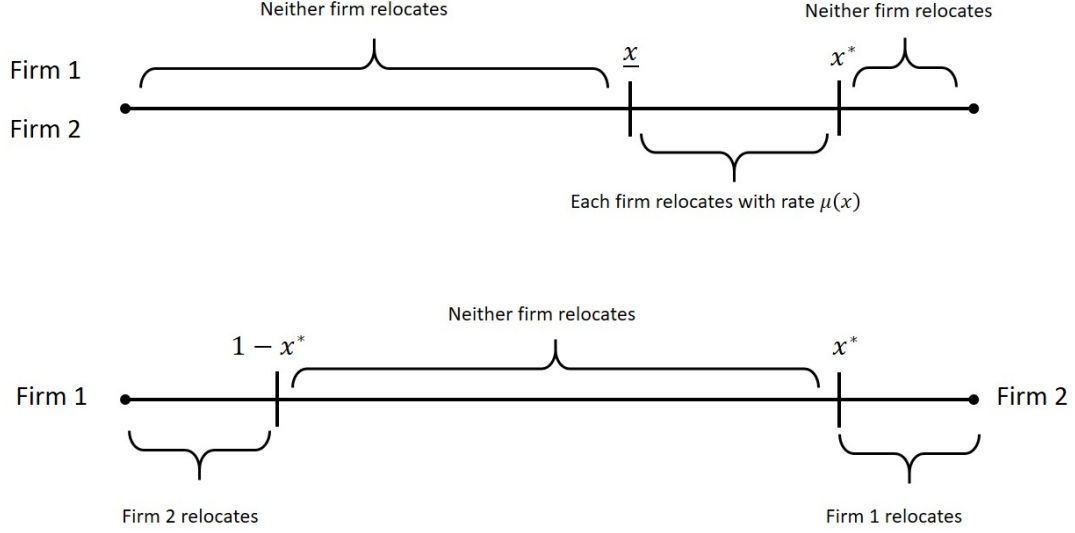


Figure 2: Equilibrium

function of its distance to consumers, x , can be written as:

$$V_d(x) = \pi_d(x)dt + e^{-rdt}\mathbb{E}[V_d(x + dx)] \quad (1)$$

$$= \frac{\pi_d(x)}{r} + \frac{1}{\lambda^2}V_d''(x)$$

$$= \frac{\pi_d(x)}{r} + \frac{4\delta^2}{9r\lambda^2} + Ae^{\lambda x} + Be^{-\lambda x} \quad \text{where } \lambda = \sqrt{\frac{2r}{\sigma^2}} \quad (2)$$

for some constants A and B to be determined.

Let \underline{x} denote the lowest x such that $\mu(x) > 0$. When firms are at the same location, and $x < \underline{x}$, a firm's value function as a function of its distance to consumers, x , can be written as:

$$V_s(x) = \pi_s dt + e^{-rdt}\mathbb{E}[V_s(x + dx)] \quad (3)$$

$$= \frac{\pi_s}{r} + Ce^{\lambda x} + De^{-\lambda x} \quad (4)$$

for some constants C and D to be determined. The reflecting boundary at $x = 0$ implies that $V_s'(0) = 0$, because at $x = 0$ the firm knows that x can only move in the positive direction.³

³See, Dixit 1993, for example.

This yields that:

$$\lambda C e^{\lambda x} - \lambda D e^{-\lambda x} = 0$$

$$C = D$$

So we can write $V_s(x) = \frac{\pi_s}{r} + C[e^{\lambda x} + e^{-\lambda x}]$ for $x < \underline{x}$.

When firms are on the same side, a firm must be indifferent between repositioning and not repositioning at \underline{x} , so that the firm is willing to mix between the two actions. So it is weakly optimal for the firm to reposition at \underline{x} . Then, we must have value-matching and smooth-pasting (e.g., Dixit 1993) at \underline{x} for optimality, which yields

$$V_d(1 - \underline{x}) - K = V_s(\underline{x}) \tag{5}$$

$$-V'_d(1 - \underline{x}) = V'_s(\underline{x}). \tag{6}$$

When firms are on different sides, a firm repositions, in the proposed equilibrium, when its distance to consumers reaches x^* . The value-matching and smooth-pasting conditions at x^* are:

$$V_d(x^*) = V_s(1 - x^*) - K \tag{7}$$

$$V'_d(x^*) = -V'_s(1 - x^*). \tag{8}$$

Finally, when the firms are on different sides and the distance to consumers decreases to $1 - x^*$, the opponent repositions in the proposed equilibrium. Then we must have value matching at $1 - x^*$ which yields⁴

$$V_d(1 - x^*) = V_s(1 - x^*) \tag{9}$$

3.3. Repositioning Rate $\mu(x)$

When firms are on the same side, let $\tilde{V}(x)$ be a firm's value function for the range of x such that $\mu(x) > 0$. The firm should be indifferent between repositioning instantly, or waiting for an infinitesimal period of length dt , during which the opponent enters with probability

⁴Note that, because this is the decision of the competitor, we do not need to have smooth pasting in this case, as there is no optimality condition for the firm.

$\mu(x)dt$. Repositioning instantly gives payoff

$$\tilde{V}(x) = V_d(1-x) - K \quad (10)$$

Waiting for period dt gives payoff

$$\begin{aligned} \tilde{V}(x) &= \pi_s dt + e^{-r dt} \mathbb{E} \left[\mu(x) dt V_d(x+dx) + (1-\mu(x)dt) \tilde{V}(x+dx) \right] \\ &= \pi_s dt + \mu(x) dt V_d(x) + (e^{-r dt} - \mu(x)dt) \left[\tilde{V}(x) + \frac{\sigma^2}{2} \tilde{V}''(x) dt \right], \end{aligned} \quad (11)$$

from which we can get

$$\mu(x) = \frac{\pi_s - r\tilde{V}(x) + \frac{\sigma^2}{2}\tilde{V}''(x)}{\tilde{V}(x) - V_d(x)} \quad (12)$$

Combining (10) and (12), and using Itô's Lemma, which gives $rV_d(1-x) = \pi_d(1-x) + \frac{\sigma^2}{2}V_d''(1-x)$, we get

$$\mu(x) = \frac{\pi_s + rK - \pi_d(1-x)}{V_d(1-x) - K - V_d(x)}. \quad (13)$$

The denominator of (13) captures the benefit of preempting the opponent. If the firm repositions at x , it gets $V_d(1-x) - K$. If the opponent repositions at x instead, it gets $V_d(x)$. Thus the denominator is the value of “beating” out the competitor in repositioning.

The numerator of (13) captures the benefit of delaying repositioning. By delaying for a length of time dt , the firm receives flow payoff π_s instead of $\pi_d(1-x)$, and reduces the cost of repositioning by rK . The equilibrium $\mu(x)$ balances the benefit of delaying repositioning with the need for preemption.

Let us now discuss the values of x for which (13) applies. We established that a firm repositions with a positive hazard rate only for some x in the range of (\underline{x}, x^*) . Note that both the numerator and the denominator of $\mu(x)$ have to be positive in order for the firm to mix at x . If the denominator is negative, then the firm strictly prefers to stay put if the opponent has some probability of repositioning. In equilibrium, both firms should not relocate at that x . Such x should be below \underline{x} . If the numerator is negative, then the firm should reposition immediately if the opponent is mixing because there is no incentive to delay. But if both firms relocate at x , a firm should deviate if $x < x^*$. Thus, $\mu(x)$ is positive only in the region (\underline{x}, x^*) .

Note that by (7) and (9), $V_d(1-x^*) - K - V_d(x^*) = 0$. Thus $\mu(x) \rightarrow \infty$ as $x \rightarrow x^{*-}$. This implies that one firm always repositions before x hits x^* . Our selection of the equilibrium

for $x > x^*$ is (mostly) payoff irrelevant. If the starting condition is not that both firms are on the same side with distance $x > x^*$ to consumers, then that condition never happens on the equilibrium path, as mentioned above. Even if the starting condition is on the same side with $x > x^*$, that condition never happens on the equilibrium path again once x goes below x^* .

3.4. Equilibrium

From (5)-(9) we can obtain (see Appendix)

$$e^{\lambda(x^*+\underline{x}-1)} = \frac{\pi_d(x^*) - \pi_s + \frac{\pi'_d(x^*)}{\lambda} + rK + \frac{4\delta^2}{9\lambda^2}}{\pi_d(1-\underline{x}) - \pi_s + \frac{\pi'_d(1-\underline{x})}{\lambda} - rK + \frac{4\delta^2}{9\lambda^2}} \quad (14)$$

and

$$e^{\lambda(x^*+\underline{x}-1)} = \frac{\pi_d(1-\underline{x}) - \pi_s - \frac{\pi'_d(1-\underline{x})}{\lambda} - rK + \frac{4\delta^2}{9\lambda^2}}{\pi_d(x^*) - \pi_s - \frac{\pi'_d(x^*)}{\lambda} + rK + \frac{4\delta^2}{9\lambda^2}}. \quad (15)$$

Equations (14) and (15) form a system of two equations from which one can obtain x^* and \underline{x} . One can then use x^* and \underline{x} to solve for A , B , and C . Plugging everything into (13) gives $\mu(x)$. This fully characterizes the equilibrium if the cost of repositioning K is not too large, such that a firm wants to reposition if consumer preferences are sufficiently far away from the firm's positioning. We state this in the following proposition.

Proposition 1. *Suppose that the cost of repositioning K is not too large. Then the symmetric market equilibrium has \underline{x} and x^* obtained as solutions of the system (14) and (15), and hazard rate of repositioning $\mu(x)$ for $x \in (\underline{x}, x^*)$ given by (13) if firms have the same positioning.*

To get sharper results, we can consider the case when $K \rightarrow 0$. In that case, we can obtain that $\underline{x}, x^* \rightarrow 1/2$, as expected. When the cost of repositioning converges to zero, firms reposition right away to the side of the market that is closer to the consumer preferences.

Perhaps more interestingly, we can obtain the speed with which \underline{x} and x^* converge to

1/2 as K converges to zero. We can obtain that as $K \rightarrow 0$ we have

$$\frac{x^* - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{9\sigma^2}{4\delta}} + \frac{1}{2}\sqrt[3]{\frac{\delta\sigma^4}{18}}K^{1/3}, \quad (16)$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{9\sigma^2}{4\delta}} - \frac{1}{2}\sqrt[3]{\frac{\delta\sigma^4}{18}}K^{1/3}, \quad (17)$$

$$\frac{x^* - \underline{x}}{K^{2/3}} \rightarrow \sqrt[3]{\frac{\delta\sigma^4}{18}}. \quad (18)$$

This yields the following result.

Proposition 2. *The thresholds x^* and \underline{x} increase infinitely fast with K for $K = 0$. For a small K , the thresholds x^* and \underline{x} increase with the variability of consumer preferences σ^2 , and decrease with the importance of the repositioning attribute, δ ; the difference ($x^* - \underline{x}$) increases with both the variability of consumer preferences σ^2 and the importance of the repositioning attribute δ . At the limit of $K = 0$, the thresholds x^* and \underline{x} are independent of the discount rate r .*

When the variability of consumer preferences, σ^2 , increases, the firms realize that it is more likely that the consumer preferences will return to where the firm is positioning. As a result, they are slower to reposition with higher x^* and \underline{x} . When the relative importance of the repositioning attribute, δ , increases, the firms become now faster to reposition, as x^* and \underline{x} decrease. This result can also be seen from the point of view of the attribute z . A lower δ means a relative increase in their degree of differentiation on attribute z , which leads to slower repositioning decisions. That is, less competition leads to less repositioning.

If firms are differentiated on the x attribute, then they are more likely to remain differentiated the greater is x^* . That is, the lower is the importance of the repositioning attribute in the consumer preferences, the more likely it is for the firms to remain differentiated.

At the limit of $K = 0$ the discount rate r does not affect the thresholds x^* and \underline{x} , as the rental cost of repositioning becomes zero.

Figure 3 presents an example of a sample path of the consumer preferences x and of the repositioning decisions of the two firms, starting from $x = .5$ with both firms positioned at $x = 0$, illustrating periods where firms are differentiated and periods where firms have the same positioning, depending on the evolution of the consumer preferences. Figures 4, 5, 6, and 7 illustrate how x^* and \underline{x} evolve with K , δ , σ^2 , and r , respectively. Note that Figure 5 illustrates that x^* and \underline{x} decrease in δ , as suggested in the proposition for K small, at

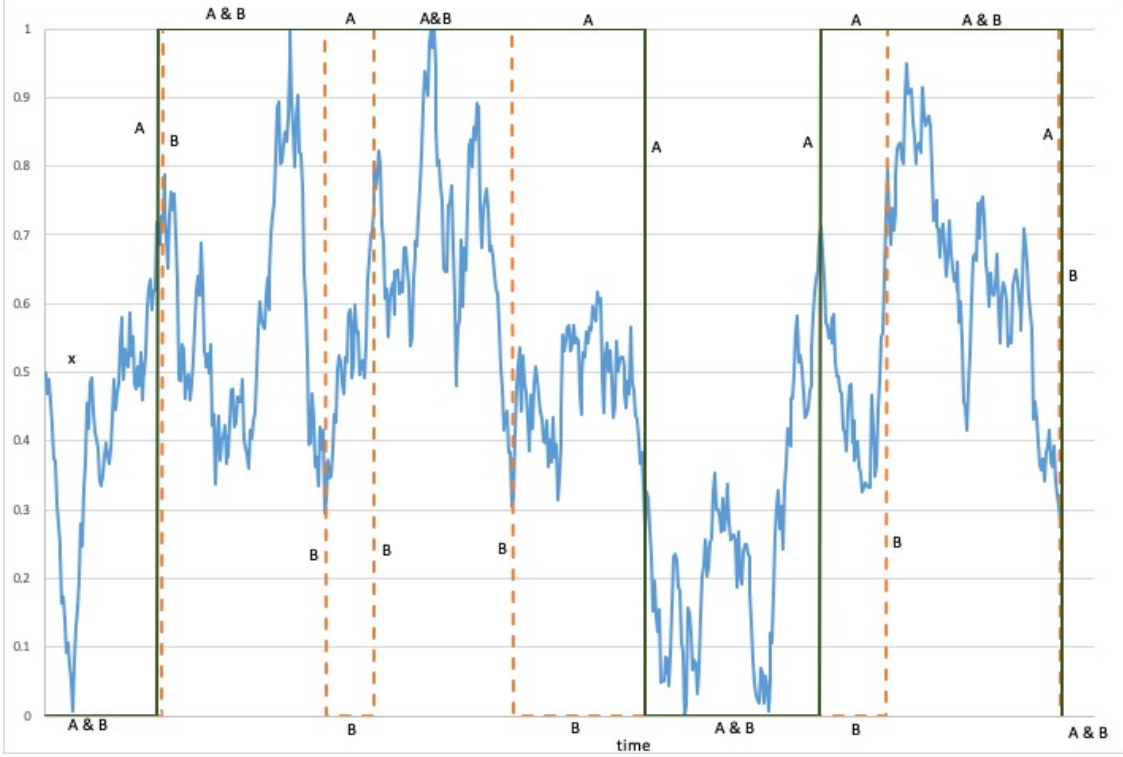


Figure 3: Example of sample path of x and of the repositioning decisions for $K = .07$, $\delta = 4$, $\sigma^2 = .2$, and $r = .1$, which yields $x^* \simeq .72$, and $\underline{x} \simeq .68$. Firm 1 is denoted as A and Firm 2 is denoted as B .

a decreasing rate. Similarly, Figure 6 illustrates how x^* and \underline{x} increase with σ^2 . Finally, Figure 7 illustrates that the effect of r on the thresholds x^* and \underline{x} is rather small, with those thresholds increasing in r . Note also that we can obtain the values of x^* and \underline{x} for $r \rightarrow 0$, and that x^* and \underline{x} are increasing in r for $K > 0$. The analysis of such a case is presented in the Appendix. In such a case, we can similarly obtain analytically that x^* , \underline{x} , and $(x^* - \underline{x})$ increase in K , σ^2 , and δ . As an example, for the case with $K = .07$, $\delta = 4$, and $\sigma^2 = .2$, we can obtain $x^* \simeq .721$ and $\underline{x} \simeq .684$ at the limit $r \rightarrow 0$.

4. EXPECTED DURATION OF DIFFERENTIATION

Consider now the duration of differentiation and co-positioning in the market. Starting from a point of differentiation we can see the expected duration going forward until when the firms choose to co-position, which occurs when the consumer preferences x reach either x^* or $1 - x^*$. Similarly, starting from a position of co-positioning we can consider the expected duration going forward until when a firm chooses to differentiate, which can start occurring when the consumer preferences x reach a distance \underline{x} from where the firms are positioned.

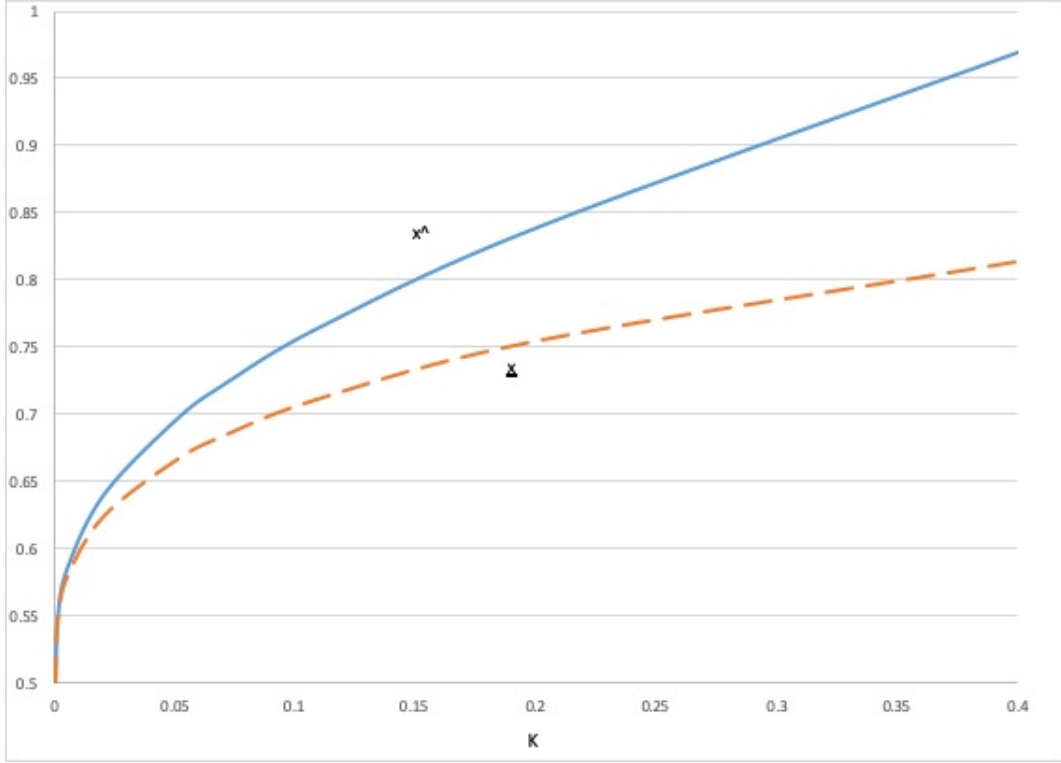


Figure 4: Evolution of x^* and \underline{x} as a function of K for $\delta = 4$, $\sigma^2 = .2$, and $r = .1$

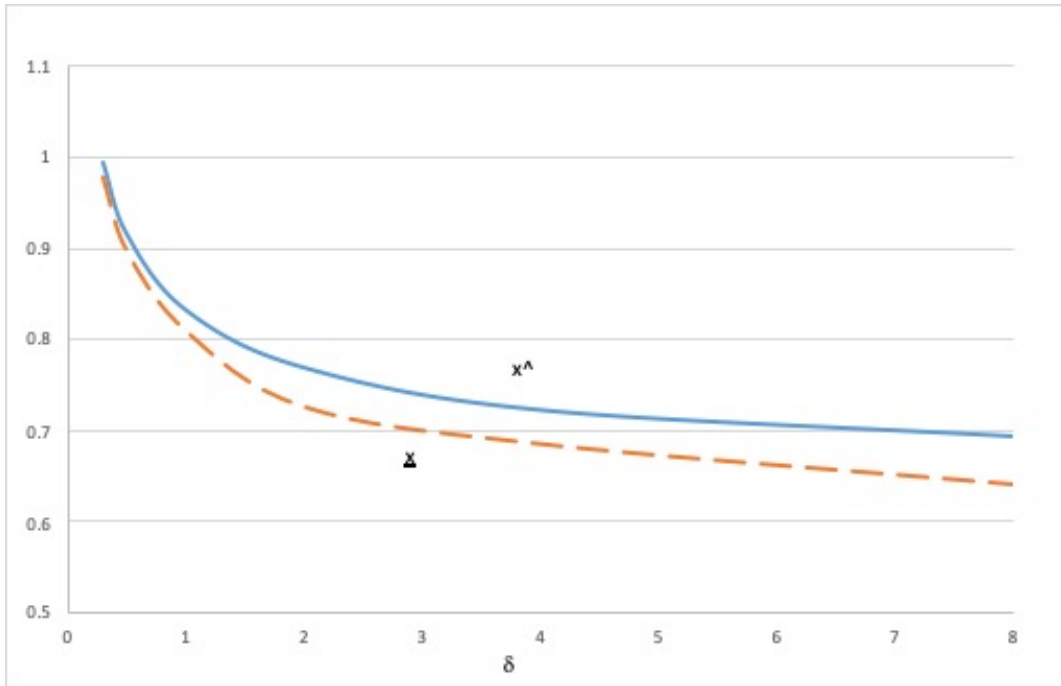


Figure 5: Evolution of x^* and \underline{x} as a function of δ for $K = .07$, $\sigma^2 = .2$ and $r = .1$

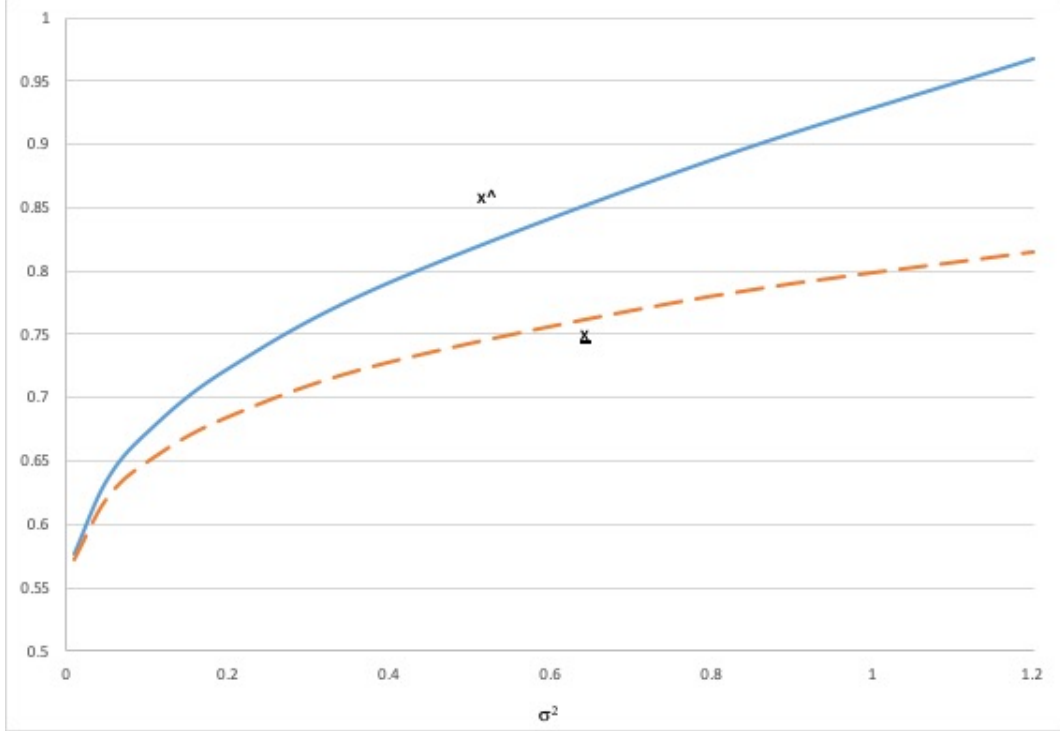


Figure 6: Evolution of x^* and \underline{x} as a function of σ^2 for $K = .07$, $\delta = 4$ and $r = .1$

Let $F_d(x)$ be the expected duration that firms continue to be differentiated if firms are differentiated and consumer preferences start at x . We know that for firms to continue to be differentiated we have that $x \in (1 - x^*, x^*)$. For a small time interval dt we can construct the equation of evolution of $F_d(x)$ as

$$F_d(x) = dt + E[F_d(x + dx)] \quad (19)$$

which results, by Itô's Lemma, in $F_d(x)'' = -2/\sigma^2$.

Using the fact that in equilibrium $F_d(x^*) = F_d(1 - x^*) = 0$, we can obtain

$$F_d(x) = \frac{1}{\sigma^2}[x(1 - x) - x^*(1 - x^*)]. \quad (20)$$

As the lowest x at which firms start differentiating is when consumers preferences reach a distance \underline{x} from where the co-positioned firms are located, we can obtain that the expected duration of differentiation is at most $F_d(\underline{x})$.

Consider now the period of time when firms are co-positioned. Let $F_s(x)$ be the expected duration that firms continue to be co-positioned if consumer preferences start at at a distance x from the firms' positioning, and one firm chooses to differentiate at the lowest possible x , \underline{x} .

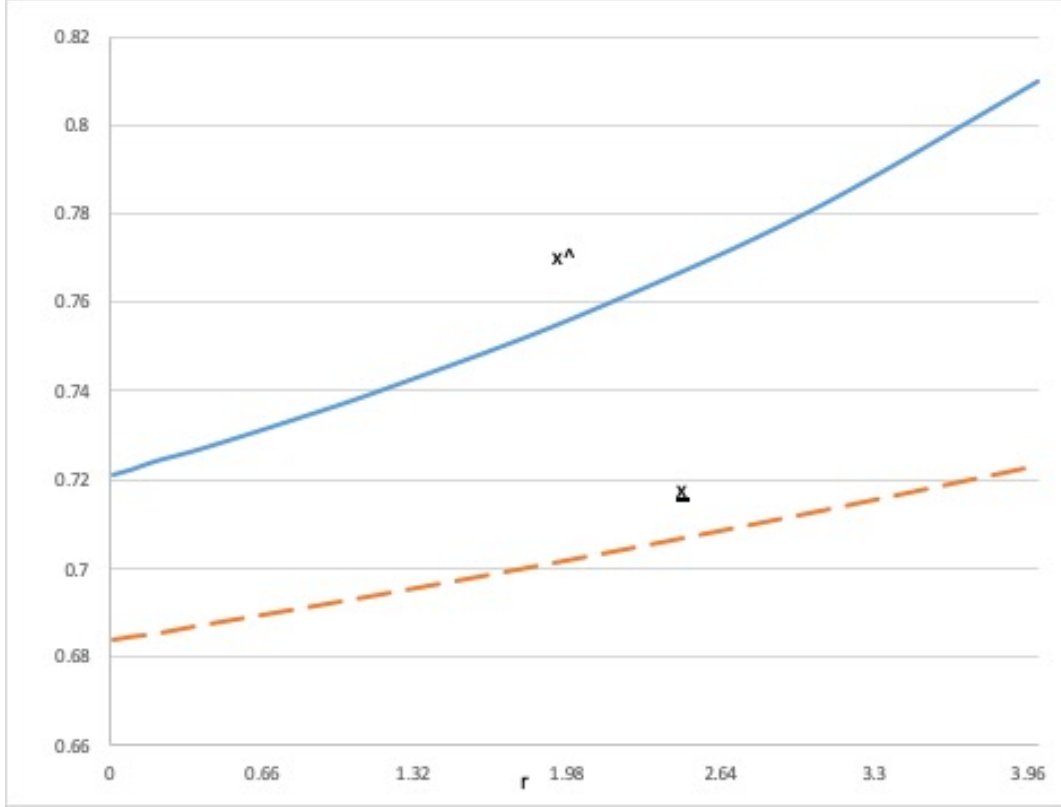


Figure 7: Evolution of x^* and \underline{x} as a function of r for $K = .07$, $\delta = 4$ and $\sigma^2 = .2$

This is the lowest x for which there is a positive hazard rate of one of the firms repositioning.⁵ We know that for firms to be co-positioned we have that $x < \underline{x}$. For a small time interval dt we can construct the equation of evolution of $F_s(x)$ as

$$F_s(x) = dt + E[F_s(x + dx)] \quad (21)$$

which results, similarly, in $F_s''(x) = -2/\sigma^2$.

By assuming the lowest possible x at which a firm can choose to reposition, \underline{x} , as the state at which a firm repositions (to get the longest possible period of differentiation) we get $F_s(\underline{x}) = 0$. Furthermore, we have $F_s'(0) = 0$, as the process x has a reflecting boundary at zero.

⁵This analysis considering the lowest expected duration of co-positioning gets sharper results and focuses on the case in which differentiation can be the highest. Considering the overall expected duration can be done numerically, and gets more complex because the consideration of the firms' mixed strategies yields a differential equation that is not solvable analytically as presented in the Appendix. Note also that in the next two sections, where we consider the cases of collusion and social welfare optimum, these computations give the exact expected duration, as in those cases one product moves with probability one at \underline{x} .

We can then obtain

$$F_s(x) = \frac{1}{\sigma^2}[\underline{x}^2 - x^2]. \quad (22)$$

As the lowest x at which firms became co-positioned is when consumers preferences reach a distance $(1 - x^*)$ from where the co-positioned firms are located, we can obtain that the highest expected duration of co-positioning is $F_s(1 - x^*)$.

For the long-run, we can get an overestimate of the fraction of time during which firms are differentiated, α_d , as $\frac{F_d(\underline{x})}{F_d(\underline{x}) + F_s(1 - x^*)}$ which yields

$$\alpha_d = x^* - \underline{x}. \quad (23)$$

From this, using (18), we can obtain that for K small, the fraction of time with differentiated firms increases in the repositioning costs K , in the variability of consumer preferences σ^2 , and in the importance of the attribute on which the firms can reposition δ . Considering the difference between the x^* and \underline{x} curves, Figures 4-7 illustrate how the fraction during which firms are differentiated depends on K , δ , σ^2 , and r . For the case when $K = .4$, $\delta = 4$, $r = .1$, and $\sigma^2 = .2$, the fraction of time when firms are differentiated is $\alpha_d = 16\%$.

5. COLLUSION

We consider now the optimal collusive behavior of the two firms and compare it with the competitive case. We analyze first the case where collusion is only on the repositioning decisions, keeping the price equilibrium as competitive, and then consider when firms collude on both repositioning and prices.

5.1. Collusive Repositioning with Competitive Pricing

As noted above, when the price equilibrium is competitive and firms are positioned in the same location, the profit for each firm is $1/2$, independent of the location of the consumer preferences. Let $\pi_s(x)$, in this Section, represent the payoff for the sum of the profits of the two firms when firms are positioned at the same location, and consumers are at a distance x from the firms' location. Then, $\pi_s(x) = 1, \forall x$.

Also as noted above, when the price equilibrium is competitive and firms are positioned at different locations, the profit of a firm when the consumers are at a distance x from that

firm is $\frac{1}{2} \left[1 + \frac{\delta(1-2x)}{3} \right]^2$. Let $\pi_d(x)$, in this section, represent the payoff for the sum of the profits of the two firms when firms are positioned in different locations, and consumers are at a distance x from one of the firms' location. Then, $\pi_d(x) = 1 + \left[\frac{\delta(1-2x)}{3} \right]^2$.

Because $\pi_d(x) \geq \pi_s(x), \forall x$, with strict inequality almost everywhere, we then have that when firms collude on repositioning and compete on prices, they will never reposition again once firms are in different locations (that is, once they are differentiated). This captures the idea that firms like to be differentiated when competing on price, as known from the static differentiation literature. However, under competitive repositioning, as obtained in the Section 3, firms are not able to remain differentiated, and can remain in the same location for some period, depending on the evolution of consumer preferences. That is, there is more repositioning in the competitive market than if firms collude on repositioning but compete on prices.

5.2. Full Collusion

Consider now the case of collusion on both repositioning and prices. In this case, when firms are positioned in the same location, and consumers are at a distance x from the firms' location, collusive pricing requires that each firm prices such that the market is fully covered (given v large enough, as assumed above), and the marginal consumer is indifferent between buying and not buying the product. This results in a price for both firms of $v - \delta x - 1/2$, and industry profits of $\pi_s(x) = v - \delta x - 1/4$.

When firms are positioned in different locations, and consumers are at a distance x from one of the firms, say a distance x from Firm 1, the collusive pricing outcome would be for firms to price such that the market is covered, and a consumer who is indifferent between purchasing the product of either firm gets a surplus of zero. Then, there is going to be a z^* , which is the distance on attribute z from Firm 1, such that that consumer gets zero surplus and is indifferent between purchasing either product. That is, Firm 1 would charge a price of $p_1 = v - \delta x - z^*$ and get a demand of z^* and Firm 2 would charge a price of $p_2 = v - \delta(1-x) - (1-z^*)$ and get a demand of $(1-z^*)$. Maximizing the industry profits on z^* yields industry profits of

$$\pi_d(x) = v - \delta(1-x) - 1 + 2 \left[\frac{2 + \delta(1-2x)}{4} \right]^2. \quad (24)$$

In this Section, let $V_s(x)$ be the net present value of industry profits when firms are

positioned at the same location, and consumer preferences are at a distance x from the firms' location, and let $V_d(x)$ be the net present value of industry profits when firms are positioned in different locations, and consumer preferences are at a distance x from the positioning of one of the firms. Then, similarly to the previous section we can obtain

$$V_s(x) = \frac{\pi_s(x)}{r} + A_s e^{\lambda x} + B_s e^{-\lambda x} \quad (25)$$

$$V_d(x) = \frac{\pi_d(x)}{r} + \frac{\pi_d''}{\lambda^2 r} + A_d e^{\lambda x} + B_d e^{-\lambda x}, \quad (26)$$

for some coefficients $A_s, B_s, A_d,$ and B_d to be determined.

The optimum is characterized by an \underline{x} and an x^* such that, when firms have the same positioning, and consumer preferences are at a distance \underline{x} , one of the firms repositions, and when firms have different positionings, and when the firm farther away from the consumer preferences is at a distance x^* from those consumer preferences, that firm repositions. Value-matching and smooth-pasting at \underline{x} requires

$$V_s(\underline{x}) = V_d(1 - \underline{x}) - K \quad (27)$$

$$V_s'(\underline{x}) = -V_d'(1 - \underline{x}). \quad (28)$$

Value-matching and smooth-pasting at x^* requires

$$V_s(1 - x^*) = V_d(x^*) + K \quad (29)$$

$$-V_s'(1 - x^*) = V_d'(x^*). \quad (30)$$

Finally, symmetry of $V_d(x)$ at $x = 1/2$ requires $V_d'(1/2) = 0$. This condition plus (27)-(30) determine \underline{x} and x^* (as described in the Appendix).

To get sharper results we can consider the case when $K \rightarrow 0$. In that case, we can obtain that $\underline{x}, x^* \rightarrow 1/2$. When the cost of repositioning converges to zero, the full collusion outcome also involves repositioning right away to the side of the market that is closer to the consumer preferences.

We can also obtain the speed of convergence when $K \rightarrow 0$. We can obtain that as $K \rightarrow 0$

we have

$$\frac{x^* - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{3\sigma^2}{2\delta}} + \frac{1}{2}\sqrt[3]{\frac{\delta\sigma^4}{12}}K^{1/3}, \quad (31)$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{3\sigma^2}{2\delta}} - \frac{1}{2}\sqrt[3]{\frac{\delta\sigma^4}{12}}K^{1/3}, \quad (32)$$

$$\frac{x^* - \underline{x}}{K^{2/3}} \rightarrow \sqrt[3]{\frac{\delta\sigma^4}{12}}. \quad (33)$$

This structure of the limits of \underline{x} and x^* is similar to the one in the competitive case considered in the previous section, and all the results stated in Proposition 2 also apply for the full collusion case considered in this section. More interestingly, we can compare the thresholds of repositioning in the competitive case with the ones in the collusion case.

Let \underline{x}_{comp} and x_{comp}^* be the values of \underline{x} and x^* , respectively, from the competitive equilibrium in the previous Section. Let \underline{x}_{coll} and x_{coll}^* be the values of \underline{x} and x^* , respectively, in the full collusion case. We can then obtain:

Proposition 3. *For K small, we obtain $1/2 < \underline{x}_{coll} < x_{coll}^* < \underline{x}_{comp} < x_{comp}^*$, and $x_{coll}^* - \underline{x}_{coll} > x_{comp}^* - \underline{x}_{comp}$.*

This shows that, under full collusion, when the cost of repositioning is small, firms reposition more frequently than in the competitive equilibrium case. In the competitive case, a firm repositions because of its private incentives to reposition. In the full collusion case, one firm repositions because of the incentives for the industry, which is able to capture the whole value of the repositioning. The whole value of the repositioning under collusive pricing is greater than the private incentives under price competition, and, therefore, the full collusion case results in more repositioning than the competitive case. Also interestingly, as $x^* - \underline{x}$ is greater in this case than in the case of competition, we have, by (23), that there is more product differentiation under collusion than in the competitive market case.

Figures 8-10 present numerically how x^* and \underline{x} compare between the collusion and competitive case and evolve as a function of K , illustrating that the results in Proposition 3 seem to hold for larger K . Similarly, Figures 11-16 illustrate how that comparison evolves as a function of δ, σ^2 , and r .

6. SOCIAL WELFARE

We consider now the optimal social welfare and compare it with the competitive and

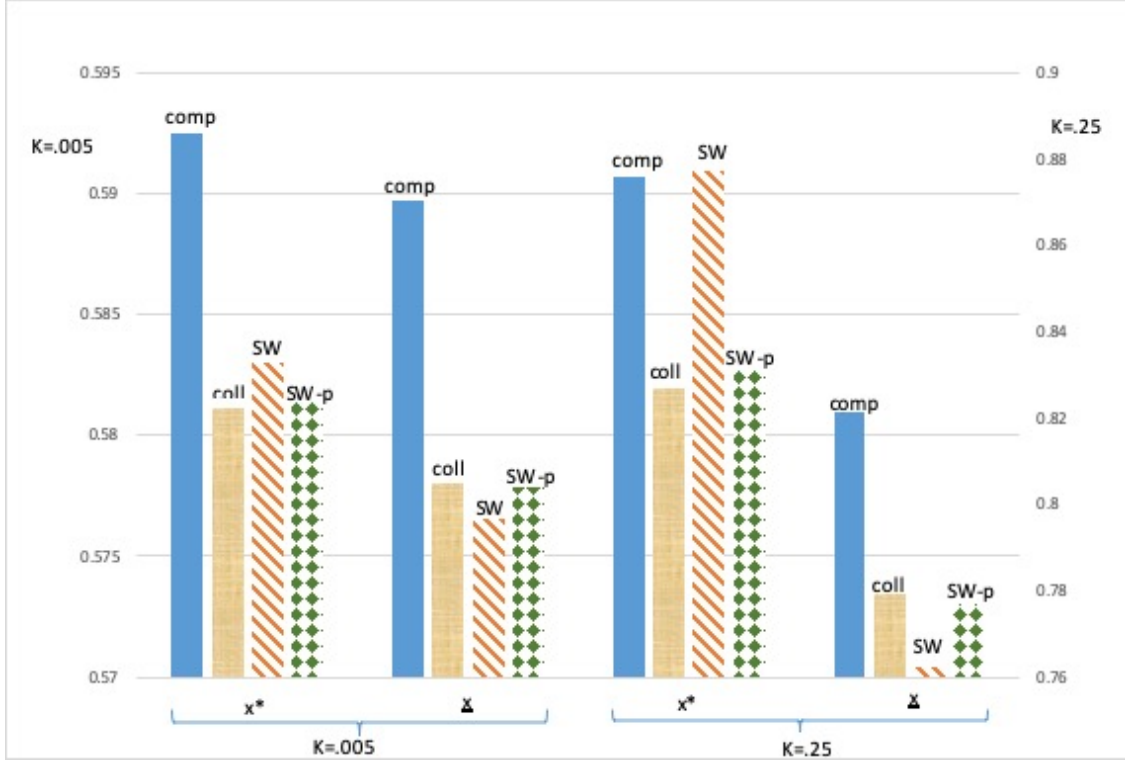


Figure 8: Comparison of x^* and \underline{x} for K low ($K = .005$) and high ($K = .25$) for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $\sigma^2 = .1$, $\delta = 1.5$, and $r = .1$. Note that the scale for the $K = .005$ case is on the left vertical axis, and the scale for the $K = .25$ case is on the right vertical axis.

collusive cases. We analyze first the case where social welfare is only optimized on the repositioning decisions, keeping the price equilibrium as competitive, and then consider the full social welfare optimum.

6.1. Social Welfare under Competitive Pricing

Consider the question of what is optimal in terms of repositioning when firms continue to price competitively. This case would be relevant when the social planner could implement some regulation on the repositioning behavior, but would have to allow firms to price competitively.

Similarly to the analysis presented above, if firms have the same positioning, then both firms would price at 1. If firms are positioned in different locations then a firm with consumers at distance x would price at $1 + \delta \frac{1-2x}{3}$. That is, competitive pricing distorts the consumers toward the less desirable firm when firms are positioned in different locations, because the less desirable firm prices at a lower price. This is going to be a force for firms not to be

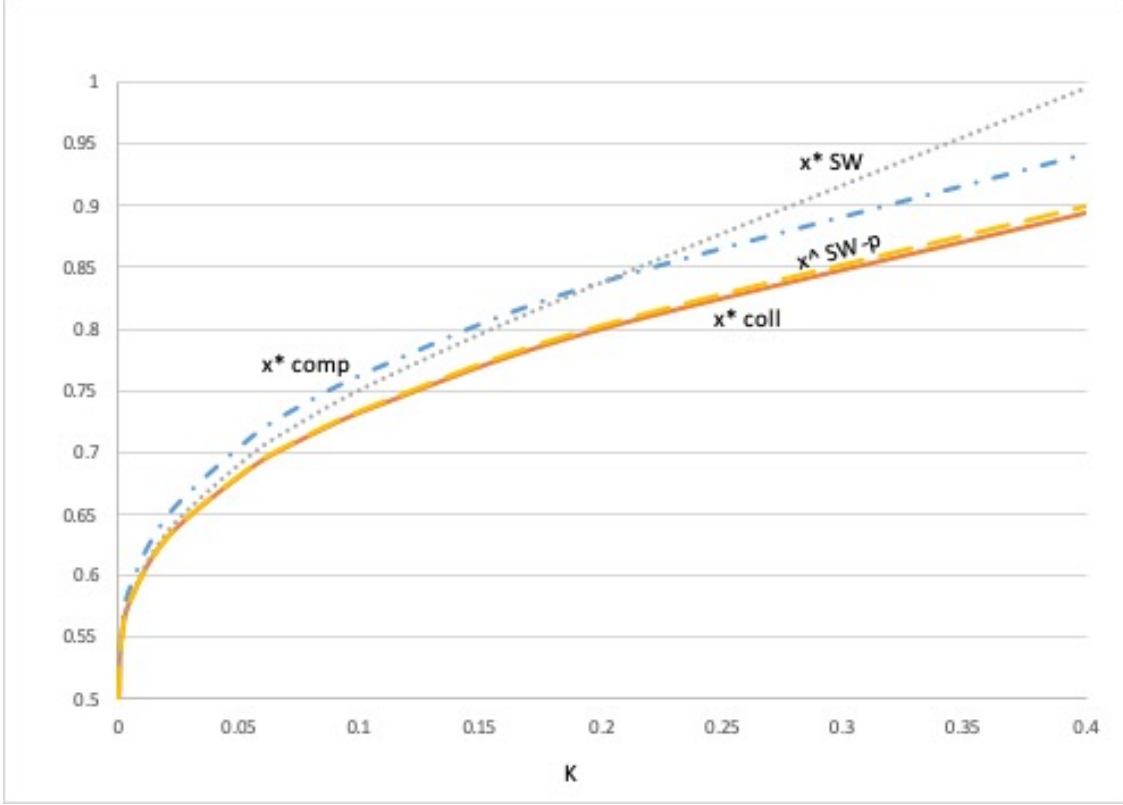


Figure 9: Evolution of x^* as a function of K for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $\sigma^2 = .1$, $\delta = 1.5$, and $r = .1$.

positioned differently when maximizing social welfare. Note also that, as we assumed v large enough, there are no market expansion effects of firms pricing lower. That is, the social welfare only has to do with the costs of repositioning and the gross utility received by the consumers from the product that is allocated to them.

Given the demand allocation that results from these prices, we can then obtain the social welfare when firms have the same positioning, with consumers at a distance x , for which we use in this subsection the notation $\pi_s(x)$, and the social welfare when firms are positioned in different locations, with consumers at distance x from one of the firms, for which we use in this subsection the notation $\pi_d(x)$. This yields

$$\pi_s(x) = v - \delta x - \frac{1}{4} \tag{34}$$

$$\pi_d(x) = v - \delta(1 - x) - \frac{7}{10} + 5 \left[\frac{9/5 + \delta(1 - 2x)}{6} \right]^2. \tag{35}$$

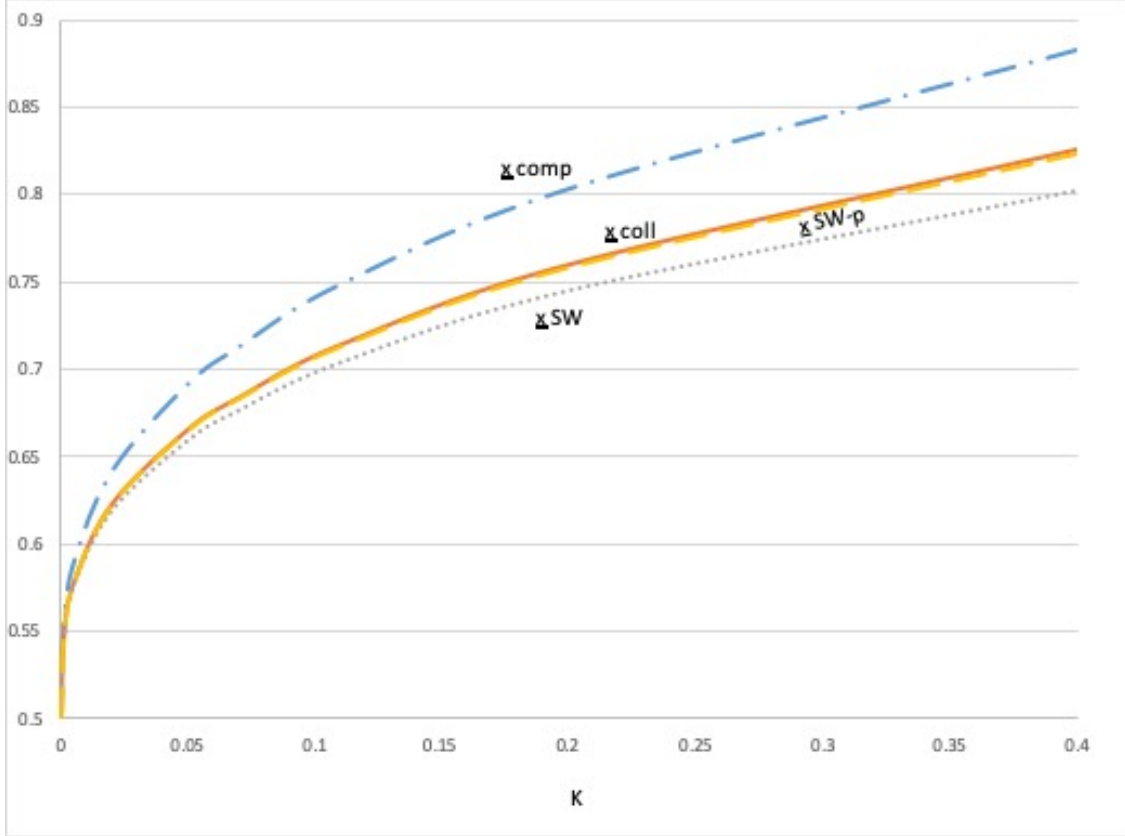


Figure 10: Evolution of \underline{x} as a function of K for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $\sigma^2 = .1$, $\delta = 1.5$, and $r = .1$.

In this Section, let $V_s(x)$ be the expected net present value of social welfare payoffs when firms are positioned at the same location, and consumer preferences are at a distance x from the firms’ location, and let $V_d(x)$ be the expected net present value of social welfare payoffs when firms are positioned in different locations, and consumer preferences are at a distance x from the positioning of one of the firms. Then we can obtain the expressions for the form of these value functions exactly as in the last section, (25) and (26), now with different functions $\pi_s(x)$ and $\pi_d(x)$, as described above.

The optimum is also characterized by an \underline{x} and an x^* (different \underline{x} and x^*) such that, when firms have the same positioning, and consumer preferences are at a distance \underline{x} , one of the firms repositions, and when firms have different positionings, and when the firm farther away from the consumer preferences is at a distance x^* from those consumer preferences, that firm repositions. Value-matching and smooth-pasting at \underline{x} and x^* require, as in the previous section, that (27)-(30) have to be satisfied. Again, symmetry of $V_d(x)$ at $x = 1/2$

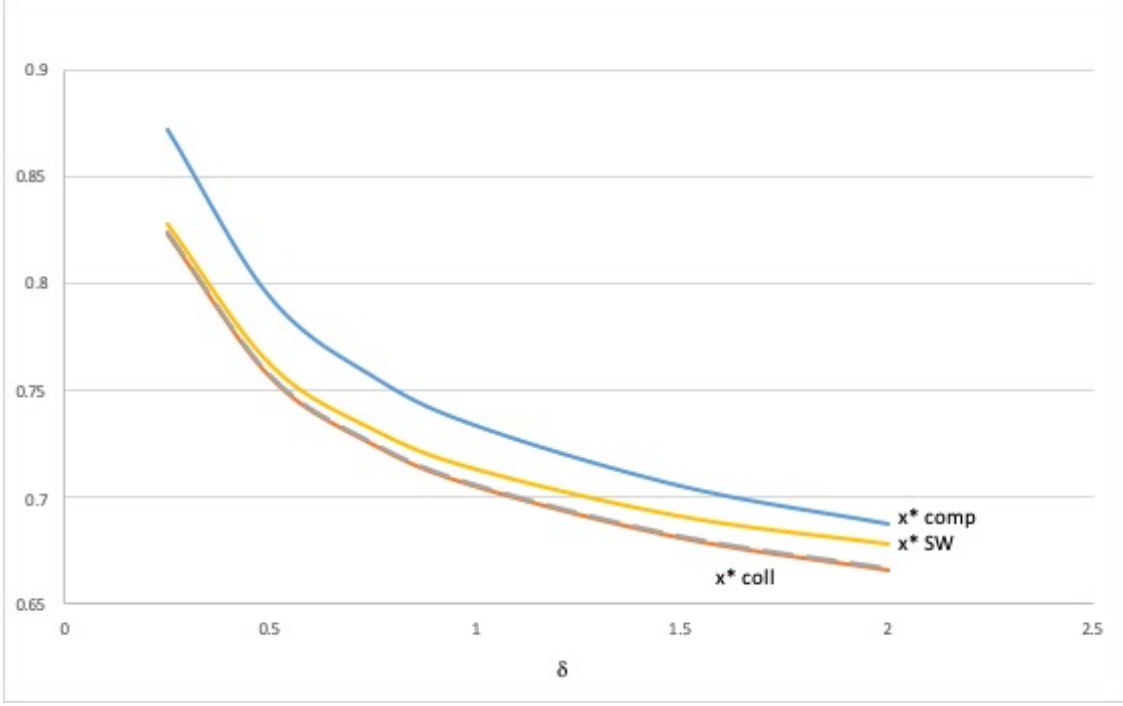


Figure 11: Evolution of x^* as a function of δ for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $\sigma^2 = .1$, $K = .05$, and $r = .1$.

requires $V'_d(1/2) = 0$. These conditions, as in the previous Section, determine \underline{x} and x^* (as described in the Appendix).

To get sharper results, we can consider the case when $K \rightarrow 0$. In that case, we can obtain again that $\underline{x}, x^* \rightarrow 1/2$. When the cost of repositioning converges to zero, the social welfare optimum subject to competitive pricing also involves repositioning right away to the side of the market that is closer to the consumer preferences.

We can also obtain that as $K \rightarrow 0$ we have

$$\frac{x^* - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{3\sigma^2}{2\delta}} + \frac{5}{18} \sqrt[3]{\frac{2\delta\sigma^4}{3}} K^{1/3}, \quad (36)$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{3\sigma^2}{2\delta}} - \frac{5}{18} \sqrt[3]{\frac{2\delta\sigma^4}{3}} K^{1/3}, \quad (37)$$

$$\frac{x^* - \underline{x}}{K^{2/3}} \rightarrow \frac{5}{9} \sqrt[3]{\frac{2\delta\sigma^4}{3}}. \quad (38)$$

This structure of the limits of \underline{x} and x^* is similar to the one in the competitive case considered in Section 3 and in the collusion case considered in the previous Section, and

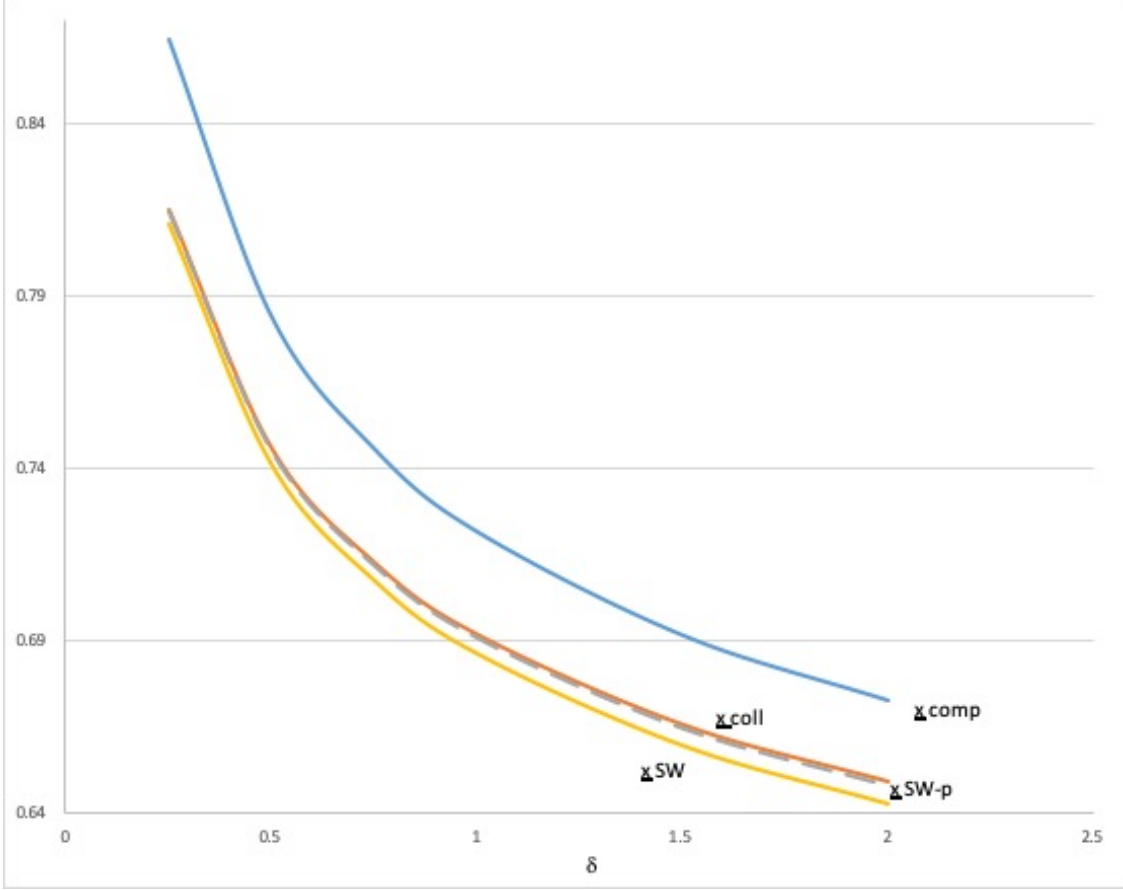


Figure 12: Evolution of \underline{x} as a function of δ for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $\sigma^2 = .1$, $K = .05$, and $r = .1$.

all the results stated in Proposition 2 also apply for the case of the social welfare optimum subject to competitive pricing considered in this Section. More interestingly, we can compare the thresholds of repositioning in the competitive case and in the collusion case with the ones in the case of the social welfare optimum subject to competitive pricing.

Let \underline{x}_{SW-p} and x_{SW-p}^* be the values of \underline{x} and x^* , respectively, in the case of the social welfare optimum subject to competitive pricing. We can then obtain:

Proposition 4. *For small K , we obtain $1/2 < \underline{x}_{SW-p} < \underline{x}_{coll} < x_{coll}^* < x_{SW-p}^* < \underline{x}_{comp} < x_{comp}^*$, and $x_{SW-p}^* - \underline{x}_{SW-p} > x_{coll}^* - \underline{x}_{coll} > x_{comp}^* - \underline{x}_{comp}$.*

This shows that, under the social optimum subject to competitive pricing, when the cost of repositioning is small, firms reposition more frequently than in the competitive equilibrium case. In the competitive case, a firm repositions because of its private incentives to reposition. In the case of the social welfare optimum subject to competitive pricing, one firm

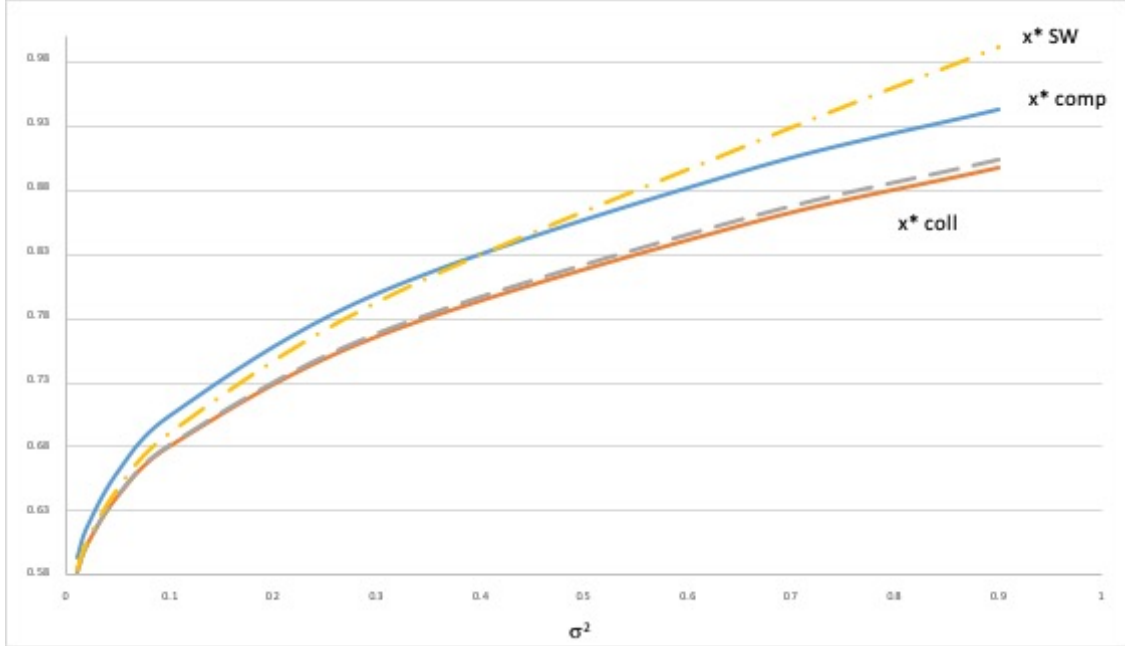


Figure 13: Evolution of x^* as a function of σ^2 for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $K = .05$, $\delta = 1.5$, and $r = .1$.

repositions because of the incentives for social welfare, which includes the whole value of the repositioning, except for the mis-allocation resulting from competitive pricing. The whole value of the repositioning for social welfare is greater than the private incentives under price competition, and, therefore, the case of the social welfare optimum subject to competitive pricing results in more repositioning than the competitive case.

The relationship of this case to the full collusion case is also interesting. First, note that the thresholds of the social optimum under competitive pricing are closer to the thresholds under full collusion than to the thresholds in the competitive market equilibrium. The difference between the thresholds of the case of the social welfare optimum subject to competitive pricing and the full collusion case are on the order of $K^{2/3}$, while the difference between the thresholds of the case of the social welfare optimum subject to competitive pricing and the competitive market equilibrium case are on the order of $K^{1/3}$. This would suggest that the outcome of full collusion is close to the social welfare optimum subject to competitive pricing, with the difference that the consumers would get a much larger surplus under the social welfare optimum subject to competitive pricing than in the full collusion case.

Second, the case of the social welfare optimum subject to competitive pricing has longer periods of firms being differentiated than in the full collusion case. In the full collusion case,

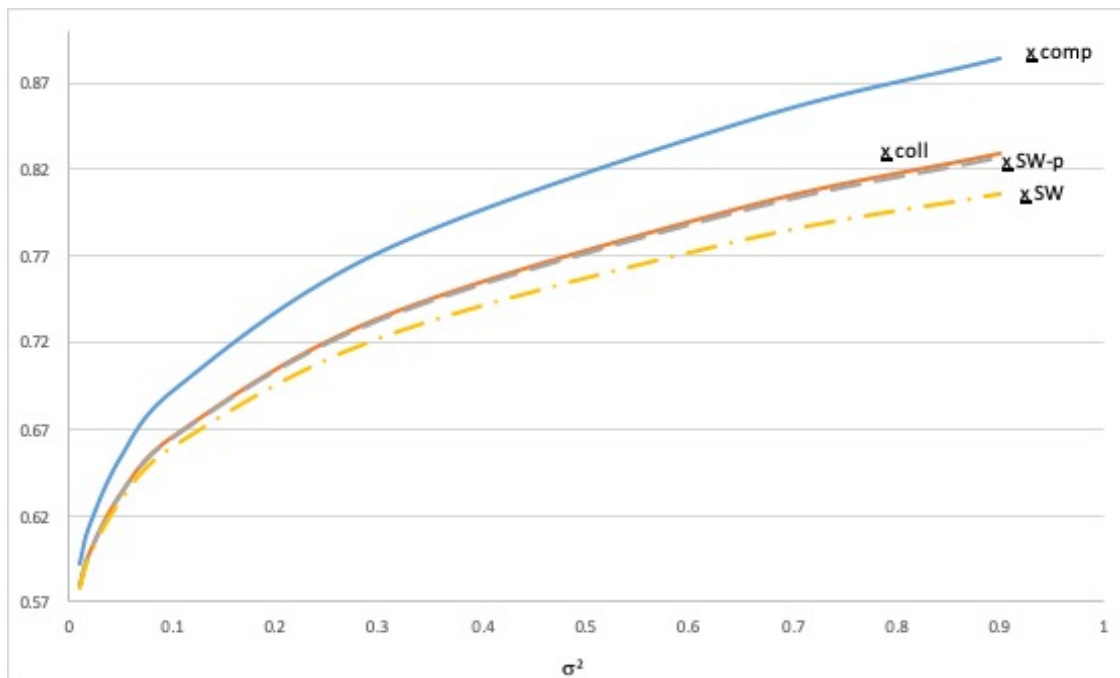


Figure 14: Evolution of \underline{x} as a function of σ^2 for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $K = .05$, $\delta = 1.5$, and $r = .1$.

the industry is not able to appropriate the utility generated to the infra-marginal consumers due to differentiation among products, which leads to the result where the firms are not differentiated enough from each other. In the case of the social welfare optimum subject to competitive pricing, the social planner includes the utility generated to the infra-marginal consumers due to the products being differentiated, and therefore keeps the products differentiated for a longer period.

Figures 8-10 present numerically how x^* and \underline{x} compare between the case of the social welfare optimum subject to competitive pricing, the collusion case, and the competitive case, and how they evolve as a function of K , illustrating that the results in Proposition 4 seem to hold for larger K . Similarly, Figures 11-16 illustrate how that comparison evolves as a function of δ , σ^2 , and r .

6.2. Social Welfare

Consider the question of what is optimal for social welfare in terms of repositioning. That is, now we consider not only the effects of the cost of repositioning on welfare, but also the effects of optimally allocating demand across the two products. Because there is no

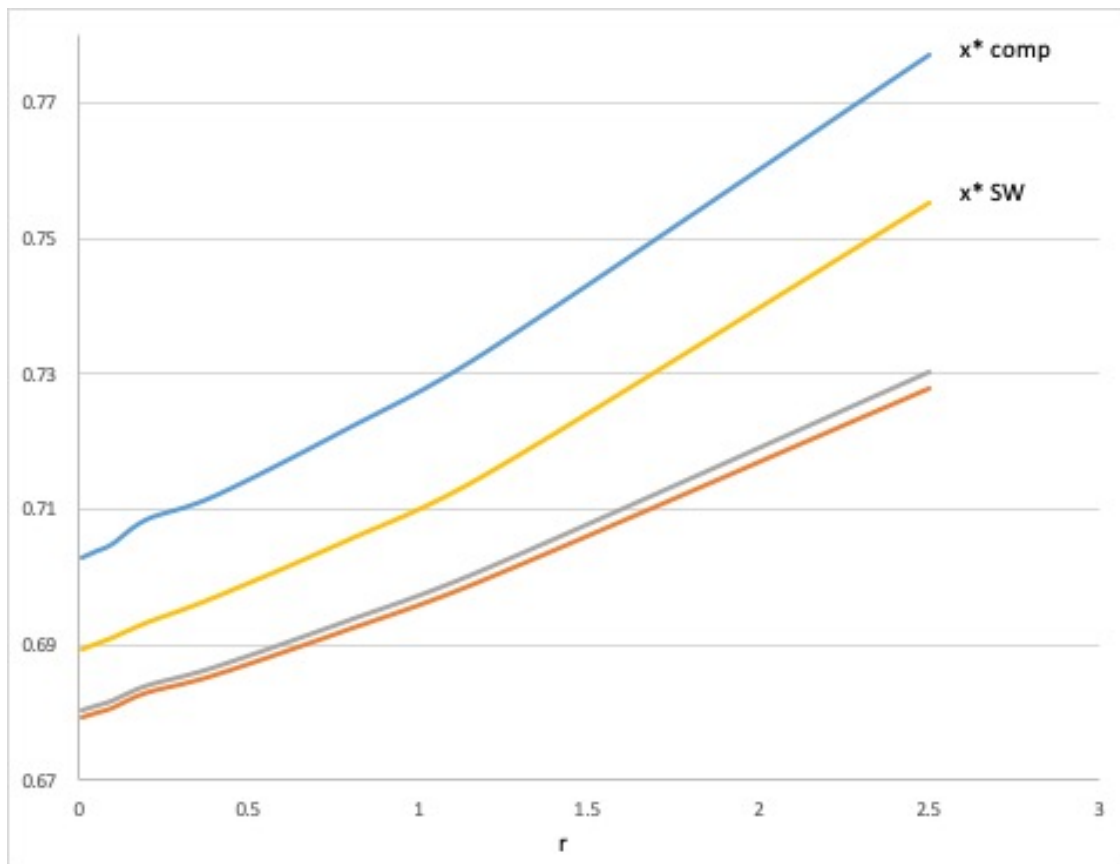


Figure 15: Evolution of x^* as a function of r for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $\sigma^2 = .1$, $\delta = 1.5$, and $K = .05$.

competitive pricing distorting consumers toward the less desirable product when products are positioned in different locations, this may then allow firms to be positioned in different locations for longer periods.

When products are positioned in the same location, demand is equally distributed between the two products. When products are positioned in different locations, the allocation of demand is just done to maximize social welfare, which results in a product located at a distance x from the consumer preferences having a demand of $\frac{1}{2} + \delta \frac{1-2x}{2}$.

Given this demand allocation, we can then obtain the social welfare when firms have the same positioning, with consumers at a distance x , for which we use in this subsection the notation $\pi_s(x)$, and the social welfare when firms are positioned in different locations, with consumers at distance x from one of the firms, for which we use in this subsection the notation $\pi_d(x)$. This yields

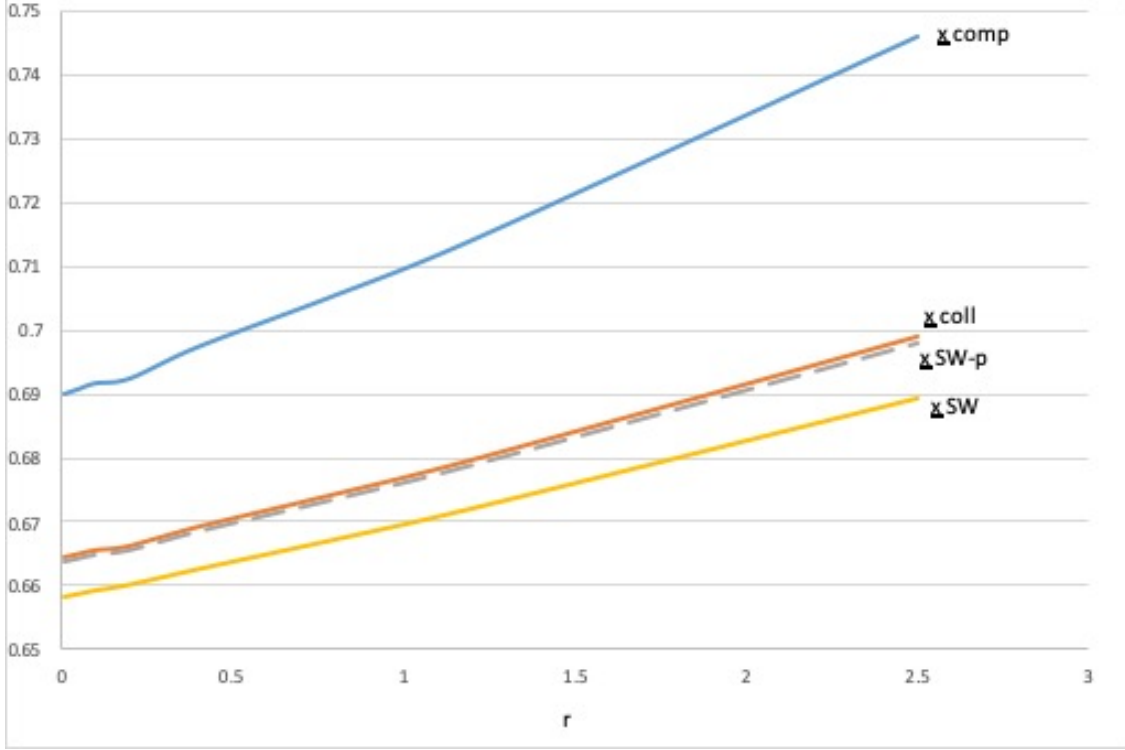


Figure 16: Evolution of \underline{x} as a function of r for the competitive (“comp”), collusion (“coll”), social welfare (“SW”), and social welfare under competitive pricing (“SW-p”) cases, for $\sigma^2 = .1$, $\delta = 1.5$, and $K = .05$.

$$\pi_s(x) = v - \delta x - \frac{1}{4} \quad (39)$$

$$\pi_d(x) = v - \delta(1 - x) - \frac{1}{2} + \left[\frac{1 + \delta(1 - 2x)}{2} \right]^2. \quad (40)$$

As noted in the previous subsection, let $V_s(x)$ be the expected net present value of social welfare payoffs when firms are positioned at the same location, and consumer preferences are at a distance x from the firms’ location, and let $V_d(x)$ be the expected net present value of social welfare payoffs when firms are positioned in different locations, and consumer preferences are at a distance x from the positioning of one of the firms. Then we can obtain the expressions for the form of these value functions exactly as in the last section, (25) and (26), now with different functions $\pi_s(x)$ and $\pi_d(x)$, as described above in this subsection.

The optimum is also characterized by an \underline{x} and an x^* (different \underline{x} and x^*) such that, when firms have the same positioning, and consumer preferences are at a distance \underline{x} , one of the firms repositions, and when firms have different positionings, and when the firm farther

away from the consumer preferences is at a distance x^* from those consumer preferences, that firm repositions. Value-matching and smooth-pasting at \underline{x} and x^* require, as in the previous section, that (27)-(30) have to be satisfied. Again, symmetry of $V_d(x)$ at $x = 1/2$ requires $V'_d(1/2) = 0$. These conditions, as in the previous Section, determine \underline{x} and x^* (as described in the Appendix).

To get sharper results we can consider the case when $K \rightarrow 0$. In that case, we can obtain again that $\underline{x}, x^* \rightarrow 1/2$. When the cost of repositioning converges to zero, the social welfare optimum subject to competitive pricing also involves repositioning right away to the side of the market that is closer to the consumer preferences.

We can obtain that as $K \rightarrow 0$ we have

$$\frac{x^* - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{3\sigma^2}{2\delta}} + \frac{1}{2} \sqrt[3]{\frac{2\delta\sigma^4}{3}} K^{1/3}, \quad (41)$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} \rightarrow \sqrt[3]{\frac{3\sigma^2}{2\delta}} - \frac{1}{2} \sqrt[3]{\frac{2\delta\sigma^4}{3}} K^{1/3}, \quad (42)$$

$$\frac{x^* - \underline{x}}{K^{2/3}} \rightarrow \sqrt[3]{\frac{2\delta\sigma^4}{3}}. \quad (43)$$

This structure of the limits of \underline{x} and x^* is similar to the one in the other cases considered, and all the results stated in Proposition 2 also apply for the case of the social welfare optimum. More interestingly, we can compare the thresholds of repositioning in the other cases with the ones in the social welfare optimum case.

Let \underline{x}_{SW} and x^*_{SW} be the values of \underline{x} and x^* , respectively, in the case of the social welfare optimum. We can then obtain.

Proposition 5. *For small K , we obtain $1/2 < \underline{x}_{SW} < \underline{x}_{SW-p} < \underline{x}_{coll} < x^*_{coll} < x^*_{SW-p} < x^*_{SW} < \underline{x}_{comp} < x^*_{comp}$, and $x^*_{SW} - \underline{x}_{SW} > x^*_{SW-p} - \underline{x}_{SW-p} > x^*_{coll} - \underline{x}_{coll} > x^*_{comp} - \underline{x}_{comp}$.*

This shows that, under the social welfare optimum, when the cost of repositioning is small, firms also reposition more frequently than in the competitive equilibrium case. In the competitive case, a firm repositions because of its private incentives to reposition. In the case of the social welfare optimum, one firm repositions because of the incentives for social welfare, which includes the whole value of the repositioning. We had already seen that this held for the case of the social welfare optimum subject to competitive pricing, and this should also obviously then hold without the constraint of competitive pricing. The whole value of the repositioning for social welfare is greater than the private incentives in the competitive market equilibrium.

The relationship of this case to the other cases considered above is also interesting. First, note that the thresholds of the social welfare optimum are closer to the thresholds under full collusion and in the case of social welfare optimum subject to competitive pricing, than to the thresholds in the competitive market equilibrium. The difference between the thresholds of the social welfare case and the cases of full collusion and social welfare subject to competitive pricing are on the order of $K^{2/3}$, while the difference between the thresholds of these cases and the competitive market equilibrium are on the order of $K^{1/3}$. This would suggest that the outcome of full collusion is close to the social welfare optimum.

Second, the case of the full social welfare optimum has longer periods of firms being differentiated than under the constraint of competitive pricing when consumer preferences. In the case of the social welfare optimum subject to competitive pricing, there was an incentive for firms to be in the same location because of the distortion in the allocation of demand due to competitive pricing. Because this incentive disappears without the constraint of competitive pricing, the optimum then has longer periods of differentiation between firms.

Figures 8-10 present numerically how x^* and \underline{x} compare between the social welfare case and the other cases, and how that comparison evolves as a function of K , illustrating that the results in Proposition 5 seem to hold for larger K . However, note that the effect of longer periods with differentiation gets stronger with larger K , such that at some point we have $x_{SW}^* > x_{comp}^*$. Similarly, Figures 11-16 illustrate how that comparison evolves as a function of δ , σ^2 , and r .

7. CONCLUSION

We have studied a model where competing firms follow consumer preferences, and choose when to reposition. We characterize how the competitive market equilibrium behaves, illustrating rich dynamics where firms are differentiated for some time, and have the same positioning during other periods. We can characterize how the market behaves depending on different market factors, showing that greater uncertainty in consumer preferences and greater importance of the repositioning attribute lead to less repositioning and more differentiation,

We compare the market equilibrium with the cases of collusion and social welfare optimization. If there is collusion on the repositioning decisions, but firms continue to price competitively, we get to a situation where there is differentiation forever. On the other hand, if there is collusion on both the repositioning decisions and pricing, or if we want to maximize

social welfare, we can obtain that there is more repositioning than in the competitive market equilibrium when the costs of repositioning are not too large. This is because the collusive firms, or the social planner, can better appropriate all the benefits of repositioning, which leads to more repositioning.

We can also obtain that under social welfare maximization there are longer periods of product differentiation than under collusion. This is because under social welfare maximization we take into account the utility of the infra-marginal consumers, which is a force towards more differentiation. Furthermore, the case of social welfare maximization subject to competitive pricing leads to shorter periods of differentiation than under unrestricted social welfare maximization, as competitive pricing creates a distortion (in favor of the less desirable product) when products are differentiated, which makes product differentiation less advantageous.

The analysis in the paper is constrained to a model of limited repositioning, only two firms, and a fully covered market. It would be interesting to consider extensions in either of these dimensions. The model allows for the firms to have only two positionings and the consumer preferences can only move within a finite segment. It would be interesting to understand what would happen if more (or a continuum of) positionings were allowed or if consumer preferences were allowed to evolve outside the finite segment. It would also be interesting to investigate what happens in a market with more than two firms. Finally, it would be interesting to study a model where the market is not necessarily fully covered, to have a more complete characterization of the welfare effects.

APPENDIX

INTERMEDIATE STEPS IN DERIVATION OF MARKET EQUILIBRIUM:

Equations (5)-(9) form the following system of equations:

$$\frac{\pi_s}{r} + Ce^{\lambda x} + Ce^{-\lambda x} = \frac{\pi_d(1-x)}{r} + \frac{4\delta^2}{9r\lambda^2} + Ae^{\lambda(1-x)} + Be^{-\lambda(1-x)} - K \quad (\text{i})$$

$$\lambda Ce^{\lambda x} - \lambda Ce^{-\lambda x} = -\frac{\pi'_d(1-x)}{r} - \lambda Ae^{\lambda(1-x)} + \lambda Be^{-\lambda(1-x)} \quad (\text{ii})$$

$$\frac{\pi_s}{r} + Ce^{\lambda(1-x^*)} + Ce^{-\lambda(1-x^*)} - K = \frac{\pi_d(x^*)}{r} + \frac{4\delta^2}{9r\lambda^2} + Ae^{\lambda x^*} + Be^{-\lambda x^*} \quad (\text{iii})$$

$$-\lambda Ce^{\lambda(1-x^*)} + \lambda Ce^{-\lambda(1-x^*)} = \frac{\pi'_d(x^*)}{r} + \lambda Ae^{\lambda x^*} - \lambda Be^{-\lambda x^*} \quad (\text{iv})$$

$$\frac{\pi_s}{r} + Ce^{\lambda(1-x^*)} + Ce^{-\lambda(1-x^*)} = \frac{\pi_d(1-x^*)}{r} + \frac{4\delta^2}{9r\lambda^2} + Ae^{\lambda(1-x^*)} + Be^{-\lambda(1-x^*)} \quad (\text{v})$$

Subtracting (ii) divided by λ from (i) gives

$$\begin{aligned} \frac{\pi_s}{r} + 2Ce^{-\lambda x} &= \frac{\pi_d(1-x)}{r} + \frac{\pi'_d(1-x)}{r\lambda} + 2Ae^{\lambda(1-x)} - K + \frac{4\delta^2}{9r\lambda^2} \\ 2Ce^{-\lambda} &= 2A + \left[\frac{\pi_d(1-x) - \pi_s}{r} + \frac{\pi'_d(1-x)}{r\lambda} - K + \frac{4\delta^2}{9r\lambda^2} \right] e^{-\lambda(1-x)} \end{aligned} \quad (\text{vi})$$

Adding (iv) divided by λ to (iii) gives:

$$\begin{aligned} \frac{\pi_s}{r} + 2Ce^{-\lambda(1-x^*)} - K &= \frac{\pi_d(x^*)}{r} + \frac{\pi'_d(x^*)}{r\lambda} + 2Ae^{\lambda x^*} + \frac{4\delta^2}{9r\lambda^2} \\ 2Ce^{-\lambda} &= 2A + \left[\frac{\pi_d(x^*) - \pi_s}{r} + \frac{\pi'_d(x^*)}{r\lambda} + K + \frac{4\delta^2}{9r\lambda^2} \right] e^{-\lambda x^*} \end{aligned} \quad (\text{vii})$$

Similarly, adding (ii) divided by λ to (i) gives

$$\begin{aligned} \frac{\pi_s}{r} + 2Ce^{\lambda x} &= \frac{\pi_d(1-x)}{r} - \frac{\pi'_d(1-x)}{r\lambda} + 2Be^{-\lambda(1-x)} - K + \frac{4\delta^2}{9r\lambda^2} \\ 2Ce^{\lambda} &= 2B + \left[\frac{\pi_d(1-x) - \pi_s}{r} - \frac{\pi'_d(1-x)}{r\lambda} - K + \frac{4\delta^2}{9r\lambda^2} \right] e^{\lambda(1-x)} \end{aligned} \quad (\text{viii})$$

Subtracting (iv) divided by λ from (iii) gives

$$\begin{aligned} \frac{\pi_s}{r} + 2Ce^{\lambda(1-x^*)} - K &= \frac{\pi_d(x^*)}{r} - \frac{\pi'_d(x^*)}{r\lambda} + 2Be^{-\lambda x^*} + \frac{4\delta^2}{9r\lambda^2} \\ 2Ce^\lambda &= 2B + \left[\frac{\pi_d(x^*)}{r} - \frac{\pi'_d(x^*)}{r\lambda} + K + \frac{4\delta^2}{9r\lambda^2} \right] e^{\lambda x^*} \end{aligned} \quad (\text{ix})$$

We can then obtain (14) from (vi) and (vii), and obtain (15) from (viii) and (ix), as they appear in the main text.

LIMIT AS $K \rightarrow 0$: Let $p^* = \frac{3+\delta-2\delta x^*}{3}$ denote price at x^* , and $\underline{p} = \frac{3-\delta+2\delta \underline{x}}{3}$ denote price at $1 - \underline{x}$. Then we have $\pi_d(x^*) = \frac{1}{2}p^{*2}$, $\pi'_d(x^*) = -\frac{2\delta}{3}p^*$, $\pi_d(1 - \underline{x}) = \frac{1}{2}\underline{p}^2$, and $\pi'_d(1 - \underline{x}) = -\frac{2\delta}{3}\underline{p}$. Furthermore, let $G = e^{\lambda(x^* + \underline{x} - 1)} = e^{\frac{3\lambda}{2\delta}(\underline{p} - p^*)}$. We can re-write equations (14) and (15) as:

$$G(\underline{p}^2 - 1) - (p^{*2} - 1) + \frac{4\delta}{3\lambda}(p^* - G\underline{p}) + \frac{8\delta^2}{9\lambda^2}(G - 1) = 2rK(1 + G) \quad (\text{x})$$

$$G(p^{*2} - 1) - (\underline{p}^2 - 1) + \frac{4\delta}{3\lambda}(Gp^* - \underline{p}) + \frac{8\delta^2}{9\lambda^2}(G - 1) = -2rK(1 + G) \quad (\text{xi})$$

Subtracting equation (xi) from (x), and dividing by $(1 + G)(p^* + \underline{p})$, we get:

$$(\underline{p} - p^*) - \frac{4\delta}{3\lambda} \frac{G - 1}{G + 1} = \frac{4rK}{p^* + \underline{p}}$$

or

$$\log G - 2 \frac{G - 1}{G + 1} = \frac{3\lambda}{2\delta} \frac{4rK}{p^* + \underline{p}}. \quad (\text{xii})$$

As $K \rightarrow 0$ in (xii), $G \rightarrow 1$, which implies $\underline{p} = p^*$, or $\underline{x} + x^* = 1$, in the limit.

Adding (x) and (xi), and dividing by $p^* - \underline{p}$, we obtain:

$$\frac{G - 1}{p^* - \underline{p}}(p^2 + p^{*2} - 2 + \frac{16\delta^2}{9\lambda^2}) + \frac{4\delta}{3\lambda}(G + 1) = 0. \quad (\text{xiii})$$

With $\frac{G-1}{p^*-\underline{p}} \rightarrow -\frac{3\lambda}{2\delta}$ as $G \rightarrow 1$ and $p^* - \underline{p} \rightarrow 0$, we obtain

$$\underline{p}^2 + p^{*2} = 2 \quad (\text{xiv})$$

which implies that, as $K \rightarrow 0$, $\underline{p}, p^* \rightarrow 1$. So, we then have that both \underline{x} and x^* approach $1/2$ in the limit.

Consider now the question of the speed of convergence. Let $y = p^* + \underline{p}$. Then, we can

write

$$x^* = \frac{1}{2} + \frac{3(2-y)}{4\delta} + \frac{1}{2\lambda} \log(G) \quad (\text{xv})$$

$$\underline{x} = \frac{1}{2} + \frac{3(y-2)}{4\delta} + \frac{1}{2\lambda} \log(G) \quad (\text{xvi})$$

$$x^* - \underline{x} = \frac{3(2-y)}{2\delta}. \quad (\text{xvii})$$

Note now that

$$\lim_{G \rightarrow 1} \frac{\log(G) - 2\frac{G-1}{G+1}}{(G-1)^3} = \frac{1}{12}. \quad (\text{xviii})$$

Then, from (xii), we can obtain

$$\lim_{K \rightarrow 0} \frac{(G-1)^3}{K} = \frac{36r\lambda}{\delta}. \quad (\text{xix})$$

Noting that $p^* = \frac{y}{2} - \frac{\delta}{3\lambda} \log(G)$ and $\underline{p} = \frac{y}{2} + \frac{\delta}{3\lambda} \log(G)$, we can obtain from (xiii) that

$$\lim_{K \rightarrow 0} \frac{y-2}{(G-1)^2} = -\frac{\delta^2 \sigma^2}{54r}. \quad (\text{xx})$$

Using this plus (xix) in (xv), (xvi), and (xvii), and using the definition of λ , we can obtain (16), (17), and (18).

LIMIT AS $r \rightarrow 0$: Noting that $\frac{\log(G)}{\sqrt{r}} = \frac{3\sqrt{2}}{2\delta\sigma}(\underline{p} - p^*)$ and using (xviii), we can obtain from (xii) that at the limit $r \rightarrow 0$ we have

$$(\underline{p} - p^*)^3(\underline{p} + p^*) = H \quad (\text{xxi})$$

where $H = 32K\delta^2\sigma^2/3$. Similarly, we can obtain from (xiii) that at the limit

$$3(p^{*2} + \underline{p}^2 - 2) = (\underline{p} - p^*)^2. \quad (\text{xxii})$$

We can then solve (xxi)-(xxii) to obtain

$$\lim_{r \rightarrow 0} x^* = \frac{1}{2} + \frac{3}{2\delta} \left[1 - \frac{1}{2} \left(\frac{H}{w^{3/2}} - \sqrt{w} \right) \right] \quad (\text{xxiii})$$

$$\lim_{r \rightarrow 0} \underline{x} = \frac{1}{2} - \frac{3}{2\delta} \left[1 - \frac{1}{2} \left(\frac{H}{w^{3/2}} + \sqrt{w} \right) \right] \quad (\text{xxiv})$$

where $w = (\underline{p} - p^*)^2$ solves $w^4 - 12w^3 + 3H^2 = 0$. For example, for $\sigma^2 \rightarrow 0$ we can obtain

$H \rightarrow 0$, which yields $w \rightarrow 0$ and $\underline{p}, p^* \rightarrow 1$, which then gives $x^*, \underline{x} \rightarrow \frac{1}{2}$.

ASYMMETRIC EQUILIBRIUM: An asymmetric equilibrium would be for Firm 1 to be the one moving first if Firms 1 and 2 have the same positioning.⁶ We look for an equilibrium where, if firms have the same positioning, Firm 1 moves to the other extreme when the consumer preferences are at a distance \underline{x} from the original positioning, and if firms have different positioning, Firm 1 moves to the other extreme if consumer preferences are at a distance x^* from its original positioning, and Firm 2 moves to the other extreme if consumer preferences are at a distance x^{**} from its original positioning.

Note first that the functional form of the value functions for both firms is still represented by (2) when firms have different positioning and by (4) when firms have the same positioning. The constants A, B, C , and D could be different for both firms. Let A, B, C , and D represent the constants for Firm 1, and let $\tilde{A}, \tilde{B}, \tilde{C}$, and \tilde{D} represent the corresponding constants for Firm 2. By the same arguments as presented in Section 3.2, we have $C = D$ and $\tilde{C} = \tilde{D}$. Finally, let V represent the value functions of Firm 1, and \tilde{V} represent the value functions of Firm 2.

Consider now the decisions of Firm 1. By the same arguments as presented in Section 3.2, the value function of Firm 1 must satisfy (5), (6), (7), and (8). By the presentation for the symmetric equilibrium case, these equations determine x^* and \underline{x} , which therefore have the same values as in the symmetric equilibrium case.

Regarding Firm 1, note finally that the value function should be continuous when the firms are in different positionings, and Firm 2 decides to move, which occurs when the consumer preferences are at a distance $(1 - x^{**})$ from Firm 1. That condition is

$$V_d(1 - x^{**}) = V_s(1 - x^{**}) \tag{xxv}$$

which replaces condition (9) in the symmetric equilibrium case.

Consider now the conditions on the value functions for Firm 2. Similarly to Firm 1, it

⁶Another potential possibility for an asymmetric equilibrium would be for Firm 1 to move first if both firms are positioned at 0, and for Firm 2 to be the one moving first if both firms are positioned at 1. Note however that such an equilibrium would not be Markov-perfect, as the payoff relevant state variable has the same value when both firms are at 0, or both firms are at 1, and the consumer preferences are at the same distance from the firms' positioning, and then the strategies by both firms have to be the same.

must be optimal for Firm 2 to reposition at x^{**} , which yields

$$\tilde{V}_d(x^{**}) = \tilde{V}_s - K \quad (\text{xxvi})$$

$$\tilde{V}'_d(x^{**}) = -\tilde{V}'_s(1 - x^{**}). \quad (\text{xxvii})$$

Similarly, Firm 2 should have value matching when Firm 1 moves at \underline{x} and $(1 - x^*)$ which yields

$$\tilde{V}_s(\underline{x}) = \tilde{V}_d(\underline{x}) \quad (\text{xxviii})$$

$$\tilde{V}_d(1 - x^*) = \tilde{V}_s(1 - x^*). \quad (\text{xxix})$$

Conditions (xxvi)-(xxix) determine uniquely x^{**} , \tilde{A} , \tilde{B} , and \tilde{C} , as we already have x^* and \underline{x} from the symmetric equilibrium conditions and the arguments above. These conditions can be written as

$$\frac{\pi_s}{r} + \tilde{C}e^{\lambda(1-x^{**})} + \tilde{C}e^{-\lambda(1-x^{**})} - K = \frac{\pi_d(x^{**})}{r} + \frac{4\delta^2}{9r\lambda^2} + \tilde{A}e^{\lambda x^{**}} + \tilde{B}e^{-\lambda x^{**}} \quad (\text{xxx})$$

$$-\lambda\tilde{C}e^{\lambda(1-x^{**})} + \lambda\tilde{C}e^{-\lambda(1-x^{**})} = \frac{\pi'_d(x^{**})}{r} + \lambda\tilde{A}e^{\lambda x^{**}} - \lambda\tilde{B}e^{-\lambda x^{**}} \quad (\text{xxxii})$$

$$\frac{\pi_s}{r} + \tilde{C}e^{\lambda\underline{x}} + \tilde{C}e^{-\lambda\underline{x}} = \frac{\pi_d(\underline{x})}{r} + \frac{4\delta^2}{9r\lambda^2} + \tilde{A}e^{\lambda\underline{x}} + \tilde{B}e^{-\lambda\underline{x}} \quad (\text{xxxiii})$$

$$\frac{\pi_s}{r} + \tilde{C}e^{\lambda(1-x^*)} + \tilde{C}e^{-\lambda(1-x^*)} = \frac{\pi_d(1-x^*)}{r} + \frac{4\delta^2}{9r\lambda^2} + \tilde{A}e^{\lambda(1-x^*)} + \tilde{B}e^{-\lambda(1-x^*)}. \quad (\text{xxxiiii})$$

Solving (xxxii) and (xxxiii) for $(\tilde{C} - \tilde{A})$ and $(\tilde{C} - \tilde{B})$, we can obtain

$$\tilde{A} = \tilde{C} - \frac{1}{rG - r/G} \left[-e^{-\lambda\underline{x}} \left(\pi_d(1-x^*) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) + e^{-\lambda(1-x^*)} \left(\pi_d(\underline{x}) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) \right] \quad (\text{xxxiv})$$

$$\tilde{B} = \tilde{C} - \frac{1}{rG - r/G} \left[e^{\lambda\underline{x}} \left(\pi_d(1-x^*) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) - e^{\lambda(1-x^*)} \left(\pi_d(\underline{x}) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) \right] \quad (\text{xxxv})$$

where $G = e^{\lambda(x^* + \underline{x} - 1)}$. Note now that dividing (xxxii) by λ , and adding it to, and subtracting

it from, (xxx), we can obtain

$$2\tilde{C}e^{-\lambda} = 2\tilde{A} + e^{-\lambda x^{**}} \left[\frac{\pi_d(x^{**}) - \pi_s}{r} + \frac{\pi'_d(x^{**})}{r\lambda} + K + \frac{4\delta^2}{9r\lambda^2} \right] \quad (\text{xxxvi})$$

$$2\tilde{C}e^{\lambda} = 2\tilde{B} + e^{\lambda x^{**}} \left[\frac{\pi_d(x^{**}) - \pi_s}{r} - \frac{\pi'_d(x^{**})}{r\lambda} + K + \frac{4\delta^2}{9r\lambda^2} \right], \quad (\text{xxxvii})$$

respectively. Using (xxxiv) in (xxxvi), and (xxxv) in (xxxvii), one obtains

$$\begin{aligned} 2\tilde{C}(e^{-\lambda} - 1) &= e^{-\lambda x^{**}} \left[\frac{\pi_d(x^{**}) - \pi_s}{r} + \frac{\pi'_d(x^{**})}{r\lambda} + K + \frac{4\delta^2}{9r\lambda^2} \right] \\ &\quad - \frac{2}{rG - r/G} \left[-e^{-\lambda \underline{x}} \left(\pi_d(1 - x^*) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) + e^{-\lambda(1-x^*)} \left(\pi_d(\underline{x}) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) \right] \end{aligned} \quad (\text{xxxviii})$$

$$\begin{aligned} 2\tilde{C}(e^{\lambda} - 1) &= e^{\lambda x^{**}} \left[\frac{\pi_d(x^{**}) - \pi_s}{r} - \frac{\pi'_d(x^{**})}{r\lambda} + K + \frac{4\delta^2}{9r\lambda^2} \right] \\ &\quad - \frac{2}{rG - r/G} \left[e^{\lambda \underline{x}} \left(\pi_d(1 - x^*) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) - e^{\lambda(1-x^*)} \left(\pi_d(\underline{x}) - \pi_s + \frac{4\delta^2}{9\lambda^2} \right) \right]. \end{aligned} \quad (\text{xxxix})$$

Dividing (xxxviii) by (xxxix), we can then obtain

$$\begin{aligned} \frac{e^{-\lambda} - 1}{e^{\lambda} - 1} &= \left\{ e^{-\lambda x^{**}} (G - 1) \left[\pi_d(x^{**}) - \pi_s + \frac{\pi'_d(x^{**})}{\lambda} + rK + \frac{4\delta^2}{9\lambda^2} \right] \right. \\ &\quad \left. - \frac{2Ge^{-\lambda \underline{x}}}{G + 1} \left[G\pi_d(\underline{x}) - \pi_d(1 - x^*) - (G - 1)\pi_s + (G - 1)\frac{4\delta^2}{9\lambda^2} \right] \right\} / \\ &\quad \left\{ e^{\lambda x^{**}} (G - 1) \left[\pi_d(x^{**}) - \pi_s - \frac{\pi'_d(x^{**})}{\lambda} + rK + \frac{4\delta^2}{9\lambda^2} \right] \right. \\ &\quad \left. - \frac{2Ge^{\lambda(1-x^*)}}{G + 1} \left[G\pi_d(1 - x^*) - \pi_d(\underline{x}) - (G - 1)\pi_s + (G - 1)\frac{4\delta^2}{9\lambda^2} \right] \right\} \quad (\text{xl}) \end{aligned}$$

which determines x^{**} given that we already have x^* and \underline{x} . For example, for $K = .07$, $\delta = 4$, $\sigma^2 = .2$, and $r = .1$, we can obtain $x^* = .721$, $\underline{x} = .684$, and $x^{**} = .730$.

EXPECTED DURATION OF FIRMS IN THE SAME LOCATION. In the main text we considered the expected duration of firms in the same location when one firm repositions when x reaches \underline{x} . We now consider this expected duration, accounting for the fact firms decide to reposition with mixed strategies with hazard rate $\mu(x)$ for $x \in (\underline{x}, x^*)$. From the analysis in the main

text, using $F'_s(0) = 0$, we have that

$$F_s(x) = a_0 - \frac{x^2}{\sigma^2} \quad (\text{xli})$$

for $x \in (0, \underline{x})$ where a_0 is a constant to be determined. For $x \in (\underline{x}, x^*)$ we have that the evolution of the expected duration $F_s(x)$ has to satisfy

$$F_s(x) = dt + [1 - \mu(x) dt]^2 E[F_x(x + dx)]. \quad (\text{xlii})$$

Using Itô's Lemma, this yields the differential equation

$$2\mu(x)F_s(x) = 1 + \frac{\sigma^2}{2}F''_s(x). \quad (\text{xliii})$$

Solving for (xliii) together with the conditions $F_s(\underline{x}^-) = F_s(\underline{x}^+)$, $F'_s(\underline{x}^-) = F'_s(\underline{x}^+)$ (smoothness of the expected duration function at \underline{x}), and $\lim_{x \rightarrow x^*} F_s(x) = 0$, we can obtain a_0 and the full characterization of $F_s(x)$ for $x \in [0, x^*)$, which can be obtained numerically.

SOCIALLY OPTIMAL AND COLLUSIVE OUTCOMES: We consider four cases for comparison: (1) socially optimal outcome, (2) collusion where firms maximize joint profit, (3) socially optimal repositioning with competitive pricing, and (4) collusive repositioning with competitive pricing. For the case of socially optimal outcome, the flow utilities are:

$$\begin{aligned} \pi_s(x) &= v - \delta x - \frac{1}{4} \\ \pi_d(x) &= v - \delta(1 - x) - \frac{1}{2} + \left[\frac{1 + \delta(1 - 2x)}{2} \right]^2 \end{aligned}$$

For the case of collusion or monopoly, the flow utilities are:

$$\begin{aligned} \pi_s(x) &= v - \delta x - \frac{1}{2} \\ \pi_d(x) &= v - \delta(1 - x) - 1 + 2 \left[\frac{2 + \delta(1 - 2x)}{4} \right]^2 \end{aligned}$$

For the case of socially optimal repositioning with competitive pricing, the flow utilities are:

$$\begin{aligned}\pi_s(x) &= v - \delta x - \frac{1}{4} \\ \pi_d(x) &= v - \delta(1-x) - \frac{7}{10} + 5 \left[\frac{9/5 + \delta(1-2x)}{6} \right]^2\end{aligned}$$

For the case of collusive repositioning with competitive pricing, the flow utilities are:

$$\begin{aligned}\pi_s(x) &= 1 \\ \pi_d(x) &= 1 + \left[\frac{\delta(1-2x)}{3} \right]^2\end{aligned}$$

Note that in the case of collusive repositioning with competitive pricing, the flow utility under different positions is always higher than the flow utility under same position. Thus if firms have different positions, they should never relocate. Below we first consider the other three cases. As noted in the text, the general solution to the decision maker's value functions has:

$$\begin{aligned}V_s(x) &= \frac{\pi_s(x)}{r} + A_s e^{\lambda x} + B_s e^{-\lambda x} \\ V_d(x) &= \frac{\pi_d(x)}{r} + \frac{\pi_d''(x)}{\lambda^2 r} + A_d e^{\lambda x} + B_d e^{-\lambda x}\end{aligned}$$

for $\lambda = \sqrt{\frac{2r}{\sigma^2}}$ and some coefficients A_s , B_s , A_d , and B_d .

When firms are on the same side, let \underline{x} denote the threshold where one firm relocates. It must then be that we have value-matching and smooth-pasting at \underline{x} , which yields

$$V_s(\underline{x}) = V_d(1 - \underline{x}) - K \tag{xliv}$$

$$V_s'(\underline{x}) = -V_d'(1 - \underline{x}) \tag{xlv}$$

When firms are on different sides, a firm repositions when its distance to consumers reaches x^* . The value-matching and smooth-pasting conditions at x^* are:

$$V_s(1 - x^*) = V_d(x^*) + K \tag{xlvii}$$

$$-V_s'(1 - x^*) = V_d'(x^*). \tag{xlviii}$$

Finally, by the symmetry of $V_d(x)$ around $x = \frac{1}{2}$, we get $V_d'(\frac{1}{2}) = 0$, which yields:

$$A_d e^\lambda = B_d \quad (\text{xlvi})$$

Equations (xlv), (xlv), (xlv), and (xlv) form the following system of equations:

$$\frac{\pi_s(\underline{x})}{r} + A_s e^{\lambda \underline{x}} + B_s e^{-\lambda \underline{x}} = \frac{\pi_d(1 - \underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} + A_d e^{\lambda(1-\underline{x})} + B_d e^{-\lambda(1-\underline{x})} - K \quad (\text{xlix})$$

$$\frac{\pi_s'(\underline{x})}{\lambda r} + A_s e^{\lambda \underline{x}} - B_s e^{-\lambda \underline{x}} = -\frac{\pi_d'(1 - \underline{x})}{\lambda r} - A_d e^{\lambda(1-\underline{x})} + B_d e^{-\lambda(1-\underline{x})} \quad (1)$$

$$\frac{\pi_s(1 - x^*)}{r} + A_s e^{\lambda(1-x^*)} + B_s e^{-\lambda(1-x^*)} = \frac{\pi_d(x^*)}{r} + \frac{\pi_d''}{r\lambda^2} + A_d e^{\lambda x^*} + B_d e^{-\lambda x^*} + K \quad (\text{li})$$

$$-\frac{\pi_s'(1 - x^*)}{\lambda r} - A_s e^{\lambda(1-x^*)} + B_s e^{-\lambda(1-x^*)} = \frac{\pi_d'(x^*)}{\lambda r} + A_d e^{\lambda x^*} - B_d e^{-\lambda x^*} \quad (\text{lii})$$

Subtracting (1) from (xlix) and using (xlvi), we get:

$$\begin{aligned} \frac{\pi_s(\underline{x})}{r} - \frac{\pi_s'(\underline{x})}{r\lambda} + 2B_s e^{-\lambda \underline{x}} &= \frac{\pi_d(1 - \underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(1 - \underline{x})}{r\lambda} + 2B_d e^{-\lambda \underline{x}} - K \\ 2B_s &= 2B_d + \left[\frac{\pi_d(1 - \underline{x}) - \pi_s(\underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(1 - \underline{x})}{r\lambda} + \frac{\pi_s'(\underline{x})}{r\lambda} - K \right] e^{\lambda \underline{x}} \end{aligned} \quad (\text{liii})$$

Adding (lii) to (li) and using (xlvi) gives:

$$\begin{aligned} \frac{\pi_s(1 - x^*)}{r} - \frac{\pi_s'(1 - x^*)}{r\lambda} + 2B_s e^{-\lambda(1-x^*)} &= \frac{\pi_d(x^*)}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(x^*)}{r\lambda} + 2B_d e^{-\lambda(1-x^*)} + K \\ 2B_s &= 2B_d + \left[\frac{\pi_d(x^*) - \pi_s(1 - x^*)}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(x^*)}{r\lambda} + \frac{\pi_s'(1 - x^*)}{r\lambda} + K \right] e^{\lambda(1-x^*)} \end{aligned} \quad (\text{liv})$$

Similarly, adding (1) to (xlix) and using equation (xlvi) gives

$$\begin{aligned} \frac{\pi_s(\underline{x})}{r} + \frac{\pi_s'(\underline{x})}{r\lambda} + 2A_s e^{\lambda \underline{x}} &= \frac{\pi_d(1 - \underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} - \frac{\pi_d'(1 - \underline{x})}{r\lambda} + 2A_d e^{\lambda \underline{x}} - K \\ 2A_s &= 2A_d + \left[\frac{\pi_d(1 - \underline{x}) - \pi_s(\underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} - \frac{\pi_d'(1 - \underline{x})}{r\lambda} - \frac{\pi_s'(\underline{x})}{r\lambda} - K \right] e^{-\lambda \underline{x}} \end{aligned} \quad (\text{lv})$$

Subtracting (lii) divided by λ from (li) gives

$$\frac{\pi_s(1-x^*)}{r} + \frac{\pi'_s(1-x^*)}{r\lambda} + 2A_s e^{\lambda(1-x^*)} = \frac{\pi_d(x^*)}{r} + \frac{\pi''_d}{r\lambda^2} - \frac{\pi'_d(x^*)}{r\lambda} + 2A_d e^{\lambda(1-x^*)} + K$$

$$2A_s = 2A_d + \left[\frac{\pi_d(x^*) - \pi_s(1-x^*)}{r} + \frac{\pi''_d}{r\lambda^2} - \frac{\pi'_d(x^*)}{r\lambda} - \frac{\pi'_s(1-x^*)}{r\lambda} + K \right] e^{-\lambda(1-x^*)} \quad (\text{lvi})$$

If we define $f(x) = \pi_d(x) - \pi_s(1-x) + \pi''_d(x)/\lambda^2$, then we can obtain from (liii) and (liv):

$$e^{\lambda(x^*+\underline{x}-1)} = \frac{f(x^*) + f'(x^*)/\lambda + rK}{f(1-\underline{x}) + f'(1-\underline{x})/\lambda - rK} \quad (\text{lvii})$$

and obtain from (lv) and (lvi):

$$e^{\lambda(x^*+\underline{x}-1)} = \frac{f(1-\underline{x}) - f'(1-\underline{x})/\lambda - rK}{f(x^*) - f'(x^*)/\lambda + rK} \quad (\text{lviii})$$

These two conditions determine the thresholds \underline{x} and x^* . Note that we can write the conditions for competitive equilibrium from equations (14) and (15) in the exact same form.

LIMIT AS $K \rightarrow 0$ FOR SOCIALLY OPTIMAL AND COLLUSIVE OUTCOMES: For the socially optimal case, we have

$$f(x) = \left[\frac{1 + \delta(1-2x)}{2} \right]^2 - \frac{1}{4} + 2\frac{\delta^2}{\lambda^2}.$$

For the collusive case, we have

$$f(x) = 2 \left[\frac{2 + \delta(1-2x)}{4} \right]^2 - \frac{1}{2} + \frac{\delta^2}{\lambda^2}.$$

For socially optimal repositioning with competitive pricing, we have

$$f(x) = 5 \left[\frac{9/5 + \delta(1-2x)}{6} \right]^2 - \frac{9}{20} + \frac{10}{9} \frac{\delta^2}{\lambda^2}.$$

More generally, we can write

$$f(x) = a \left[\frac{b + \delta(1-2x)}{c} \right]^2 - a \left(\frac{b}{c} \right)^2 + 8 \frac{a}{c^2} \frac{\delta^2}{\lambda^2}. \quad (\text{lix})$$

For future use, note that in the competitive case $a = 1/2$ and $b = c = 3$.

Let $p^* = \frac{b+\delta(1-2x^*)}{c}$, and $\underline{p} = \frac{b+\delta(2\underline{x}-1)}{c}$. Furthermore, let $G = e^{\lambda(x^*+\underline{x}-1)} = e^{\frac{c\lambda}{2\delta}(p-p^*)}$. We

can re-write (lvii) and (lviii) as:

$$G \left[\underline{p}^2 - \left(\frac{b}{c} \right)^2 \right] - \left[p^{*2} - \left(\frac{b}{c} \right)^2 \right] + \frac{4\delta}{c\lambda}(p^* - G\underline{p}) + \frac{8\delta^2}{c^2\lambda^2}(G - 1) = rK(1 + G)/a \quad (\text{lx})$$

$$G \left[p^{*2} - \left(\frac{b}{c} \right)^2 \right] - \left[\underline{p}^2 - \left(\frac{b}{c} \right)^2 \right] + \frac{4\delta}{c\lambda}(Gp^* - \underline{p}) + \frac{8\delta^2}{c^2\lambda^2}(G - 1) = -rK(1 + G)/a \quad (\text{lxi})$$

Subtracting (lxi) from (lx), and dividing by $(1 + G)(p^* + \underline{p})$, we get:

$$(\underline{p} - p^*) - \frac{4\delta}{c\lambda} \frac{G - 1}{G + 1} = \frac{2rK}{a(p^* + \underline{p})}$$

or

$$\log G - 2 \frac{G - 1}{G + 1} = \frac{c\lambda}{a\delta} \frac{rK}{p^* + \underline{p}} \quad (\text{lxii})$$

As $K \rightarrow 0$ in (lxii), $G \rightarrow 1$, which implies $\underline{p} = p^*$, or $\underline{x} + x^* = 1$, in the limit.

Adding (lx) and (lxi), and dividing by $p^* - \underline{p}$, we obtain:

$$\frac{G - 1}{p^* - \underline{p}} \left[\underline{p}^2 + p^{*2} - 2 \left(\frac{b}{c} \right)^2 + \frac{16\delta^2}{c^2\lambda^2} \right] + \frac{4\delta}{c\lambda}(G + 1) = 0 \quad (\text{lxiii})$$

With $\frac{G-1}{p^*-\underline{p}} \rightarrow -\frac{c\lambda}{2\delta}$ as $G \rightarrow 1$ and $p^* - \underline{p} \rightarrow 0$, we obtain

$$\underline{p}^2 + p^{*2} = 2 \left(\frac{b}{c} \right)^2 \quad (\text{lxiv})$$

which implies that as $K \rightarrow 0$, $\underline{p}, p^* \rightarrow \frac{b}{c}$. So, we then have that both \underline{x} and x^* approach 1/2 in the limit.

Consider now the question of the speed of convergence. Let $y = p^* + \underline{p}$. Then, we can write

$$x^* = \frac{1}{2} + \frac{2b - cy}{4\delta} + \frac{1}{2\lambda} \log(G) \quad (\text{lxv})$$

$$\underline{x} = \frac{1}{2} + \frac{cy - 2b}{4\delta} + \frac{1}{2\lambda} \log(G) \quad (\text{lxvi})$$

$$x^* - \underline{x} = \frac{2b - cy}{2\delta} \quad (\text{lxvii})$$

Again we have from (xviii) that

$$\lim_{G \rightarrow 1} \frac{\log(G) - 2\frac{G-1}{G+1}}{(G-1)^3} = \frac{1}{12}$$

Then, from (lxii), we can obtain

$$\lim_{K \rightarrow 0} \frac{(G-1)^3}{K} = \frac{6c^2 r \lambda}{ab \delta} \quad (\text{lxviii})$$

Noting that $p^* = \frac{y}{2} - \frac{\delta}{c\lambda} \log(G)$ and $\underline{p} = \frac{y}{2} + \frac{\delta}{c\lambda} \log(G)$ we can obtain from (lxiii) that

$$\lim_{K \rightarrow 0} \frac{y - 2\left(\frac{b}{c}\right)}{(G-1)^2} = -\frac{1}{6bc} \frac{\delta^2 \sigma^2}{r}. \quad (\text{lxix})$$

Using this plus (lxviii), we can write (16), (17), and (18) for this general case, as $K \rightarrow 0$, as:

$$\frac{x^* - 1/2}{K^{1/3}} = \left(\frac{c^2}{ab}\right)^{1/3} \frac{1}{2\lambda} \left(\frac{6r\lambda}{\delta}\right)^{1/3} + \left(\frac{c^2}{ab}\right)^{2/3} \frac{\delta\sigma^2}{24br} \left(\frac{6r\lambda}{\delta}\right)^{2/3} K^{1/3} \quad (\text{lxx})$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} = \left(\frac{c^2}{ab}\right)^{1/3} \frac{1}{2\lambda} \left(\frac{6r\lambda}{\delta}\right)^{1/3} - \left(\frac{c^2}{ab}\right)^{2/3} \frac{\delta\sigma^2}{24br} \left(\frac{6r\lambda}{\delta}\right)^{2/3} K^{1/3} \quad (\text{lxxi})$$

$$(\text{lxxii})$$

which implies that

$$\lim_{K \rightarrow 0} \frac{x^* - 1/2}{K^{1/3}} = \left(\frac{c^2}{ab}\right)^{1/3} \left(\frac{3\sigma^2}{8\delta}\right)^{1/3} \quad (\text{lxxiii})$$

$$\lim_{K \rightarrow 0} \frac{x^* - \underline{x}}{K^{2/3}} = \frac{1}{2b} \left(\frac{c^2}{ab}\right)^{2/3} \left(\frac{\delta\sigma^4}{3}\right)^{1/3}. \quad (\text{lxxiv})$$

Equations (lxxiii) and (lxxiv) then allow us to compare the thresholds across different scenarios for K small. Let us use \underline{x}_{comp} and x^*_{comp} to denote the thresholds from the competitive equilibrium. Let \underline{x}_{SW} and x^*_{SW} denote the thresholds from the socially optimal outcome. Let \underline{x}_{coll} and x^*_{coll} denote the thresholds from the collusive outcome. And let \underline{x}_{SW-p} and x^*_{SW-p} denote the thresholds from the case of socially optimal repositioning with competitive price. We have that, for K close to zero:

$$\begin{aligned} \underline{x}_{SW} &< \underline{x}_{SW-p} < \underline{x}_{coll} < \underline{x}_{comp} \\ x^*_{coll} &< x^*_{SW-p} < x^*_{SW} < x^*_{comp} \end{aligned}$$

This comparison shows that competition leads to less repositioning compared to both the socially optimal and the collusion case.

REFERENCES

- BONATTI, A., G. CISTERNAS, AND J. TOIKKA (2017), “Dynamic Oligopoly with Incomplete Information,” *Review of Economic Studies*, **84**, 503-546.
- BUDD, C., C. HARRIS, AND J. VICKERS (1993), “A Model of the Evolution of Duopoly: Does the Asymmetry between Firms Tend to Increase or Decrease?,” *Review of Economic Studies*, **60**, 543-573.
- CABRAL, L.M.B., AND M. RIORDAN (1994), “The Learning Curve, Market Dominance, and Predatory Pricing,” *Econometrica*, **62(5)**, 1115-1140.
- DIXIT, A. (1993), *The Art of Smooth Pasting*, Harwood Academic Publishers: Chur, Switzerland.
- ERICKSON, R., AND A. PAKES (1995), “Markov-Perfect Industry Dynamics: A Framework for Empirical Work,” *Review of Economic Studies*, **62**, 53-82.
- HARRIS, C., AND J. VICKERS (1987), “Racing with Uncertainty,” *Review of Economic Studies*, **54**, 1-21.
- HAUSER, J.R. (1988), “Competitive Price and Positioning Strategies,” *Marketing Science*, **7**, 76-91.
- HOTELLING, H. (1929), “Stability in Competition,” *The Economic Journal*, **39(153)**, 41-57.
- JEZIORSKI, P. (2014), “Estimation of Cost Synergies from Mergers: Application to the U.S. Radio,” *RAND Journal of Economics*, **45**, 816-846.
- LAUGA, D.O., AND E. OFEK (2011), “Product Positioning in a Two-Dimensional Vertical Differentiation Model: The Role of Quality Costs,” *Marketing Science*, **30**, 903-923.
- MAGILL, M.J.P., AND G.M. CONSTANTINIDES (1976), “Portfolio Selection with Transaction Costs,” *Journal of Economic Theory*, **13**, 245-263.
- MOORTHY, K.S. (1988), “Product and Price Competition in a Duopoly,” *Marketing Science*, **7**, 141-168.
- OFEK, E., AND M. SARVARY (2003), “R&D, Marketing, and the Success of Next-Generation Products,” *Marketing Science*, **22**, 355-370.
- SAYMAN, S., S.J. HOCH, AND J.S. RAJU (2002), “Positioning of Store Brands,” *Marketing Science*, **21**, 378-397.
- SCARF, H. (1959), “The Optimality of (S,s) Policies in the Dynamic Inventory Problem,” in Arrow, K., S. Karlin, and P. Suppes (eds.), *Mathematical Methods in the Social Sciences*, Stanford University Press.

- SHESHINSKI, E., AND Y. WEISS (1983), “Optimum Pricing Policy under Stochastic Inflation,” *Review of Economic Studies*, **L**, 513-529.
- SPENCE, M. (1975), “Monopoly, Quality, and Regulation,” *The Bell Journal of Economics*, **6**, 417-429.
- SWEETING, A. (2013), “Dynamic Product Positioning in Differentiated Product Markets: The Effect of Fees for Musical Performance Rights on the Commercial Radio Industry,” *Econometrica*, **81**, 1763-1803.
- VILLAS-BOAS, J.M. (1992), “Dynamic Duopolies with Non-Convex Adjustment Costs,” *Economics Letters*, **39**, 391-394.
- VILLAS-BOAS, J.M. (2018), “A Dynamic Model of Repositioning,” *Marketing Science*, **37(2)**, 279-293.
- WERNERFELT, B. (1989), “From Critical Resources to Corporate Strategy,” *Journal of General Management*, **14**, 4-12.